CSCI 2510 - Problem set 6

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Problem 14.5

Given a graph $G = (V, E)$ with nonnegative vertex weights $c(v)$ and a valid k-coloring of G, We find a $2(1 - \frac{1}{k})$-approximation to vertex cover as follows. Use the simplex method to find an extreme point optimal solution $\{x_v^\star\}_V$ to the LP relaxation of the problem (LP 14.2 in the textbook). By Theorem 14.5 this solution is half integral. Consider the sets $V_0, V_{\frac{1}{2}}$ and V_1 of vertices whose corresponding variables have values $0, \frac{1}{2}$ and 1, respectively. For $i = 0, \frac{1}{2}, 1$, let w_i denote the total weight of vertices in V_i, $w_i = \sum_{v \in V_i} c(v)$. Since G is k-colorable, there must be a color c such that $\sum_{v \in V_{\frac{1}{2}} \text{ and } v\text{'s color is } c} c(v) \geq \frac{w_{\frac{1}{2}}}{k}$. Pick as the vertex cover all vertices of V_1, and all vertices that are not c-colored in $V_{\frac{1}{2}}$.

Lemma 1. *The solution thus obtained is feasible.*

Proof. Assume the contrary and let $e = (u, v)$ be an edge that was not covered by the solution. Consider the optimal solution to the relaxation $x^\star$. Since $x^\star$ is half integral and feasible there are only two options:

1. at least one of u, v belongs to V_1, in which case our algorithm would choose either v or u to the vertex cover.

2. both v and u belong to $V_{\frac{1}{2}}$. Since v and u are incident to the same edge, they must have different colors. Hence, at least one of them is not c-colored and will be picked to the vertex cover.

In both cases we obtained a contradiction, hence all edges must be covered by the vertex cover □

Lemma 2. *The approximation guarantee of the algorithm is* $2(1 - \frac{1}{k})$.

Proof. Since $x^\star$ is an optimal solution to the relaxed problem, it is a lower bound on OPT $OPT \geq w(x^\star)$. We can express the cost of $x^\star$ as $w(x^\star) = \frac{1}{2}w_{\frac{1}{2}} + w_1$. By choice of the color c, the cost of the solution obtained by our algorithm is at most $w_1 + w_{\frac{1}{2}}(1 - \frac{1}{k}) \leq 2(1 - \frac{1}{k})w(x^\star)$. Combining the two inequalities we get that the weight of our solution is at most $2(1 - \frac{1}{k})OPT$. □