Problem Set 7

Problem 16.5

Remark 16.6 notes that randomly chosing between the 1/2 factor algorithm's assignment and the $1-1/e$ factor algorithm's assignment means that each boolean variable x_i has probability $\frac{1}{4}+\frac{1}{2}y_i^*$ of being set to True, although the events are not independent. This observation suggests the following conjecture.

Conjecture. *Setting x_i to True independently with probability $g(y_i^*) = \frac{1}{4}+\frac{1}{2}y_i^*$ yields a 3/4 factor algorithm for MAX-SAT.*

Proof. The proof is nearly identical to that of the $1-1/e$ factor algorithm for MAX-SAT. First we lower bound the probability that an arbitrary clause c will be satisfied by the algorithm's assignment. Without loss of generality, replace each negated literal in clause c with its nonnegated counterpart (updating the LP accordingly), and rename the variables in c such that $c = (x_1 \vee \cdots \vee x_k)$. The former is possible because of a property of our choice of g: the probability that each modified literal will satisfy c remains unchanged. Observe that originally:

$$\mathbf{Pr}[\bar{x}_i \text{ is True}] = 1 - \mathbf{Pr}[x_i \text{ is True}] = 1 - g(\dot{y}_i^*),$$

where $\dot{y}_i^*$ denotes the solution to the original LP. After replacing $\bar{x}_i$ with x_i, we have:

$$\mathbf{Pr}[x_i \text{ is True}] = g(y_i^*) = g(1-\dot{y}_i^*),$$

since the modified LP just replaces $\dot{y}_i$ with $1-y_i$. The identity $1-g(x) = g(1-x)$ is easily verified for our choice of g.

The probability that c is satisfied is then given by:

$$1 - \prod_{i=1}^{k}(1-g(y_i^*)) \geq 1 - \left(\frac{\sum_{i=1}^{k}(1-g(y_i^*))}{k}\right)^k \geq 1 - \left(\frac{3}{4} - \frac{z_c^*/2}{k}\right)^k$$

Let

$$\gamma_k = 1 - \left(\frac{3}{4} - \frac{1/2}{k}\right)^k,$$

and

$$g(z) = 1 - \left(\frac{3}{4} - \frac{z/2}{k}\right)^k.$$

Since $g(z)$ is concave, $g(z) \geq \gamma_k z$ for $0 \leq z \leq 1$. So $\mathbf{Pr}[c \text{ is satisfied}] \geq \gamma_k z_c^*$. The rest of the argument is exactly the same as for the $1-1/e$ factor algorithm: the linearity of expectation and the objective function of the LP relaxation together show that the algorithm is a factor γ_k. Finally we show that $\gamma_k \geq 3/4$ for all $k \geq 1$:

$$1 - \left(\frac{3}{4} - \frac{1/2}{k}\right)^k = 1 - \left(\frac{3}{4}\right)^k \left(1 - \frac{2/3}{k}\right)^k \geq 1 - \left(\frac{3}{4}\right)^k e^{-2/3} \geq 3/4$$

for $k \geq 3$. For $k=1$ and $k=2$, $\gamma_k = 3/4$ exactly. □