Homework 7

Problem 16.2

Show that the following is a factor 1/2 algorithm for MAX-SAT. Let τ be an arbitrary truth assignment and τ' be its complement. Compute the weight of clauses satisfied by τ and τ', then output the better assignment.

Let C be the set of all clauses

Let $c \subseteq C$ be the set of clauses fulfilled by τ

Let $\overline{c} = C \setminus c$ be the set of clauses not fulfilled by τ

Let $c' \subseteq C$ be the set of clauses fulfilled by τ'

Let $w(c)$ denote the weight of the set of clauses c

Let $w(\tau)$ denote the weight of the set of clauses fulfilled by τ

First, show that either c or $\overline{c}$ is at least a $\frac{1}{2}$-factor approximation for MAX-SAT. $c \cup \overline{c} = C \implies w(c) + w(\overline{c}) = w(C)$ because the weight of a set of clauses is the sum of the weights of the clauses. $w(C) \geq OPT$ forms a trivial bound on OPT since MAX-SAT can do no better than total success. Therefore, $w(c) + w(\overline{c}) = w(C) \implies \big(w(c) \geq \frac{1}{2}w(C) \geq \frac{1}{2}OPT\big) \vee \big(w(\overline{c}) \geq \frac{1}{2}w(C) \geq \frac{1}{2}OPT\big)$.

Next, bound the result given by the assignment τ. By definition, $\tau \implies c$. Therefore, $w(c) \geq \frac{1}{2}OPT \implies \big(\tau \implies w(\tau) \geq \frac{1}{2}OPT\big)$.

Bound the result given by the assignment τ'. $\forall \delta \in \overline{c}, \forall t \in \delta, \tau \implies \neg t$, therefore, $\tau' \implies t \implies \delta \implies \overline{c}$. Therefore, $c' \supseteq \overline{c} \implies w(c') \geq w(\overline{c})$, which would mean $w(c') \geq w(\overline{c}) \geq \frac{1}{2}OPT \implies \big(\tau' \implies w(\tau') \geq \frac{1}{2}OPT\big)$.

Since either $w(c)$ or $w(\overline{c})$ exceeds $\frac{1}{2}OPT$ and $w(\tau) = w(c), w(\tau') \geq w(\overline{c})$ then at least one of τ, τ' form a $\frac{1}{2}$-factor approximation of OPT.