CS2510 Homework

Feng-Hao Liu

Department of Computer Science, Brown University

1 Problem 18.8

Given two different edges e, e' in the tree, let r be the root of the tree and e is deeper than e', i.e. there is a path $P = (r, v_0, v_1, \ldots v_i, v_{i+1}, \ldots, v_j, v_{j+1})$, where $e' = (v_i, v_{i+1})$, $e = (v_j, v_{j+1})$ (That is the definition of "deeper").

Prove Outline:

1. The problem asks to prove that if both edges are in D in the algorithm, then e is added before or at the same round with e'.
2. Let's prove it by contradiction: suppose both edges are added in D and e' is added before e, then there is a contradiction.
3. This is equivalent to say: suppose in a particular step, e' is added to D but e is not, then in the later steps e will never be added in D. This will cause the contradiction.

Suppose at some step, we pushed a flow on $P' = Path(s_k, t_k)$ and $e' = (v_i, v_{i+1})$ is saturated, but $e = (v_j, v_{j+1})$ is not. Then we know $dist(P', r) \leq i$ since $dist(r, v_i) = i$ and the distance between r and the path must be less equal than that between r and any node on the path. So in the later steps, the only way that e is added in D is that there exists a path $P'' = (s_\ell, t_\ell)$ such that (1) $dist(P'', r) \leq dist(P', r) \leq i$, (2) we push a flow on P'' and it saturates e. Note: the algorithm considers the paths from a non-increasing order of the distance to the root.

Claim: Any path $\tilde{P}$ with (1) $dist(\tilde{P}, r) \leq i$ (2) $e \in \tilde{P}$ must cross through e'.

Consider the other way: suppose there is a path P^* with $dist(P^*, r) \leq i$ and $e \in P^*$ but it does not cross through e'. Let $u \in P^*$ be the node that is closest to r. Note: the (unique) path from r to u can not meet v_i in the middle. That is because it takes i edges from r to v_i but $dist(r, u) \leq i$. Now, we have a path from r to u (not containing v_i), a path from r to v_j (containing v_i), and a path from v_j to u (not containing v_i). This will form a cycle, and thus a contradiction to the definition of a tree!

By the claim we know P'' must cross through e'. However, since e' is added into D, which implies it has been saturated. Therefore, we could not push any flow into this path. This means e will never be saturated, and will never be added into D. This completes the proof. $\square$