CS 251 - Problem 18.9

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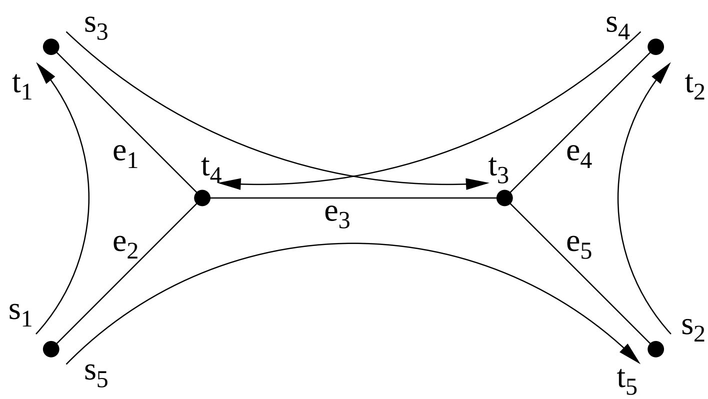
We are solving the following problem:



If we let d_i be a variable indicating whether we have decided to cut edge e_i, then we can write the multicut LP relaxation for this problem as follows:

$$\begin{aligned} \text{minimize } & d_1 + d_2 + d_3 + d_4 + d_5 \\ \text{s.t. } & d_1 + d_2 \geq 1 \\ & d_4 + d_5 \geq 1 \\ & d_1 + d_3 \geq 1 \\ & d_3 + d_4 \geq 1 \\ & d_2 + d_3 + d_5 \geq 1 \end{aligned}$$

I claim that the optimal fractional solution to this problem is:

$$\begin{aligned} d_1 = d_4 &= 2/3 \\ d_2 = d_3 = d_5 &= 1/3 \end{aligned}$$

This solution gives a total value of 7/3, and clearly satisfies all of the constraints. To show that it is optimal, we consider the dual of the LP:

$$\begin{aligned}\text{maximize } & f_1 + f_2 + f_3 + f_4 + f_5 \\ \text{s.t. } & f_1 + f_3 \leq 1 \\ & f_2 + f_4 \leq 1 \\ & f_1 + f_5 \leq 1 \\ & f_2 + f_5 \leq 1 \\ & f_3 + f_4 + f_5 \leq 1\end{aligned}$$

Consider the following solution to the dual multicommodity flow problem:

$$\begin{aligned} f_1 = f_2 &= 2/3 \\ f_3 = f_4 = f_5 &= 1/3 \end{aligned}$$

All of the constraints are satisfied, so this is a feasible solution. And it has value 7/3. Therefore, since we have a feasible solution in the dual, we know that for the primal, $OPT \geq 7/3$. But we have already found a solution of value 7/3, so we know that the previously stated solution is an optimal one.

Now we wish to find a minimal integral multicut. Because the above LP is a relaxation of the integer program for multicut, we know that the $OPT_{integral} \geq OPT_{LP} = 7/3$. The optimal integral solution must be an integer, so in addition, we know that $OPT_{int} \geq 3$.

Consider the cut $C = \{e_1, e_3, e_4\}$. Because all edges have unit weight, we know that this cut has cost 3. So if this solution is feasible, then it is an optimal integral solution to the multicut problem. It is easy to see that it is feasible, because once those edges have been removed, the only edges remaining are e_2 and e_5. These edges aren't even connected, and all of the source-sink paths have length at least two, so we have an optimal integral multicut.