# Homework X

## Problem 22.2

The metric Steiner forest problem is identical to the Steiner forest problem, with the exception that there is a restriction such that the edge costs satisfy the triangle inequality.

We can show that there is an approximation factor preserving reduction from the Steiner tree problem by constructing the following reduction.

Given as input to the Steiner forest problem, a graph $G = (V, E)$, with some edge weights $c_e$, and a collection of disjoint subsets of $V$, $S_1, \ldots, S_k$, we construct a new complete graph $G'$. The vertices in $G'$ are the same as they are in $G$, so $V' = V$. The cost of an edge $(u, v)$ is equal to the cost of the shortest path from $u$ to $v$. The sets of Steiner vertices, $S_1, \ldots, S_k$ are the same.

The cost for any edge in $E'$ is less than or equal to than its cost in $E$. Therefore, the optimal solution to the metric Steiner forest problem is no worse than an optimal solution of the original problem.

These edge weights also do conform to the triangle inequality. We can prove this by contradiction. Assume there are three vertices $u, v, w$ that do not conform to the triangle inequality, so $c_{(u,v)} > c_{(u,w)} + c_{(w,v)}$. This implies that to get from $u$ to $v$, there is a shorter path in $G$ than the one that was used to construct $(u, v)$ in $G'$ that passes through $w$. However, since our algorithm says that we are constructing $(u, v)$ using the shortest $u - v$ path in $G$, this is a contradiction.

Then, given a Steiner forest $F'$ within $G'$, we can obtain, in polynomial time, a Steiner forest in the original $G$ that has at most the same cost. Replace each edge $(u, v)$ in $F'$ with the path that corresponds to it in $G$. The cost of each path, by definition, must cost at most the same amount as the edge in the original tree, so this conversion will produce a set of edges that cost that at most as much as the cost of $F'$.

This construction does include $S_1, \ldots, S_k$, as any two vertices $u, v$ connected in $F'$ will still be connected in the $F$ we have created. This $F$ may contain cycles, however, so our algorithm must remove edges to make sure there are none. This only lowers the cost of the $F'$ we find that removing these edges does not increase the approximation factor of the forest we have found, so the approximation factor is the same for this reduction of Steiner forest to metric Steiner forest.

There is no loss of generality in requiring that the edge costs conform to the triangle inequality, because any graph that has edge weights that do not abide by the triangle inequality can be reduced to a graph that does have edge costs that do conform to the triangle inequality in polynomial time and in a way that does preserve the approximation factor.