Homework 9

Problem 22.6

(a)

Consider two cases: one case in which R_2 does not have any unsatisfied sets at time t, and another case in which R_2 *does* have an unsatisfied set at time t.

Case 1: If R_2 does not have any unsatisfied sets at time t, then by definition all of its vertices in S_1 are connected by a tight path. Since the vertices in S_1 for R_1 are a subset of the vertices in S_1 for R_2, any vertices $u, v \in S_1$ connected by a tight path in R_1 at time t must also be connected by a tight path in R_2.

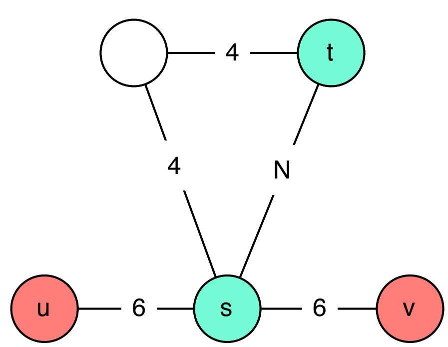
Case 2: Next, consider the case where R_2 has an unsatisfied set at time t. For this case, we will prove a claim stronger than the original one: if R_2 has an unsatisfied set, then any edge $e \in E$ that is tight in R_1 is also tight in R_2. If this claim is true, then any tight path connecting $u, v \in S_1$ for R_1 must also connect $u, v \in S_1$ for R_2.

Assume the claim is false, so that it's possible for there to be a tight edge in R_1 but not in R_2. The only way for an edge to be tight in R_1 but not in R_2 is if, at some point between 0 and t, one of the nodes touching edge e was in an active set in R_1 but not in R_2.

Initially, R_1 and R_2 contain the same active sets, except that R_2 has one additional active set. So any node initially in an active set for R_1 is also in an active set for R_2. This means that any edge that feels a dual after the first step in R_1 also feels a dual in R_2.

Because R_2 has an unsatisfied set, none of its active sets will stop growing. Thus, although the additional vertex in R_2 can cause two active sets to be merged together, it cannot cause any vertex to no longer be in an active set. Thus, at any point between 0 and t, any vertex in an active set in R_1 is also in an active set in R_2, so it can't be possible for there to be a tight edge in R_1 but not in R_2. Therefore, there can't be a tight path for $u, v \in S_1$ for R_1 but not R_2.

(b)



The figure gives an example of run R_1, where u and v are in S_1, s and t are in S_2, and $k = 2$. Let N in the figure be an arbitrarily large number. At time $t = 3$, u and v will be connected by a tight path. However, if run R_2 is the same as run R_1 except that the white vertex was also in S_2, then all vertices in S_2 would be connected at time $t = 2$, and neither edge (u, s) nor (s, v) would be tight at time $t = 3$. Thus, we have a counterexample for the previous claim for a case when $k > 1$.