CSCI 2510 - Problem set 9

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Problem 22.8

1. In the point-to-point connection problem we are given a graph $G = (V, E)$ with non-negative cost on the edges $c(e)$ and two disjoint sets S,T of the same cardinality. We wish to find a minimum cost subgraph connecting each vertex in S to a unique vertex in T.

   First we note that the formulation of the problem is misleading. IT seems as if the roles of S and T are not symmetric, but in fact the requirement that there is a path from *each* vertex in S to a unique vertex of T implies that there is a path from each vertex of T to a unique vertex of S.

   Consider the function $f : 2^V \rightarrow \{0, 1\}$ such that for a subset C of the V, $f(C) = 1$ if the number of vertices from S in C (i.e., $|S \cap C|$) is different from the number of vertices from T in C. Let G' be a subgraph of G. We claim that G' satisfies the point-to-point connectivity requirement iff for all C the number of edges in G' crossing the cut $(C, \bar{C})$ is at least $f(C)$. Assume there is a path in G' connecting each node in S to a unique vertex in T, and let C contain a different number of vertices from S and T. By the symmetry of S and T we may assume, without loss of generality, that there are more vertices of S in C than vertices of T. Then there exists a path from one of those vertices, say s, to a vertex of T not in C. Hence the number of edges of G' crossing the cut $(C, \bar{C})$ is at least 1. For the other direction, assume the point-to-point connectivity property does not hold. Then, there must be m vertices in S, say $s_1, \ldots, s_m$ that have paths in G' to at most $m - 1$ distinct vertices in T. Consider the set C of vertices of the connected components in G' that contain $s_1 \ldots s_m$. This set contains at least m vertices of S and at most $m - 1$ vertices of T, so $f(C) = 1$, but C has no edges leaving it.

   Next, we show that f is a *proper* function as defined in problem 22.7. By problem 22.7, this would imply that the 2-approximation for the Steiner forest problem also applies to the point-to-point connectivity problem. Indeed, by our definition of f, $f(V) = 0$. Next, since the cardinalities of S and T are the same, if C contains a different number of vertices from S and from T, so does $\bar{C}$. Hence, $f(C) = f(\bar{C})$. Finally, if A and B are disjoint subsets, and $f(A \cup B) = 1$ then $A \cup B$ contains a different number of vertices from S and T. Hence, at least one of A and B contains a different number of vertices from S and T, so at least one of $f(A)$ and $F(B)$ is 1, which shows the third and last property of a *proper* function.

2. To obtain a 2-approximation for the case where we only require that each vertex of S be connected to some vertex of T, we reduce the problem to Steiner tree. Add a dummy vertex τ to G and connect it with zero weight edges to all vertices of T. Adding these edges does not change the value of an optimal solution since any path that starts at some $s \in S$ and uses one of the new edges must first pass through some vertex $t \in T$. hence s

is connected to some $t \in T$ even without the new edge. Consider the Steiner tree problem on this graph with $S \cup T$ as the required vertices. Any solution to our original problem can be converted with zero cost to a Steiner tree by adding all of the new edges. On the other hand, any Steiner tree can be converted to a solution to our original problem by deleting all new edges. Therefore, a 2-approximation to the Steiner tree problem is also a 2-approximation to the relaxed point-to-point connectivity problem.