Homework 1

Problem 2.2

We are trying to find a locally optimal solution to is the cardinality maximum cut problem, which, given an undirected graph G, asks for a partion of V into two subsets, A and B, such that the number of edges running between vertices in subsets A and B is maximized.

Consider the algorithm that finds a locally optimal solution upder the flip operation. It starts with an arbitrary partition of V, and then looks for a vertex which, when flipped from one subset to the other, improves the number of edges between A and B, and then flips it. It does this until no vertex qualifies for a flip, at which point it terminates.

We want to show that this algorithm is a factor 1/2 approximation algorithm. Given any splitting of V into subsets A and B, we call E_{AB} the subset of edges going from subset A to subset B. We call E_{AA}, E_{BB} the subset of edges going from A-A and B-B respectively. We know that, regardless of the partion:

$|E_{AB}| \leq |E|$, therefore, we have an upper bound on OPT of
$OPT \leq |E|$

To prove that it is a factor 1/2 approximation algorithm, if the current configuration of vertices doesn't meet this criteria, then:
$|E_{AB}| < \frac{|E|}{2}$
Since $|E| = |E_{AB}| + |E_{AA}| + |E_{BB}|$, we have
$2 * |E_{AB}| < |E_{AB}| + |E_{AA}| + |E_{BB}|$
$|E_{AA}| + |E_{BB}| - |E_{AB}| > 0$

We can describe the size of these sets of edges as one half the sum of incident edges over all vertices:
$\frac{1}{2} * \sum_v e_{vAA} + e_{vBB} - e_{vAB} > 0$

$\sum_v |e_{vAA}| + |e_{vBB}| - |e_{vAB}| > 0$

Therefore, for some v_0,
$|e_{v_0AA}| + |e_{v_0BB}| - |e_{v_0AB}| > 0$

Since the size of any set is an integer, we know:
$|e_{v_0AA}| + |e_{v_0BB}| - |e_{v_0AB}| \geq 1$

By flipping v_0 from one set to the other, all incident edges contained in one subset will become incident edges spanning between the two subsets, and all incident edges spanning between the two subsets will become contained in one subset. Therefore, the size of the cut increases by:

$$|e_{v_0AA}| + |e_{v_0BB}| - |e_{v_0AB}| \geq 1$$

Therefore, given a configuration that is less that 1/2 optimal, we can always find a vertex v_0 such that flipping it improves the size of the cut by at least 1. The graph G contains a polynomial number of edges-$O(n^2)$, so we know that $OPT \leq n^2$. The initial configuration starts off at a value of at least 0, so the algorithm is guaranteed take at most $O(n^2)$ flips before reaching a locally optimal state. Since finding a vertex to flip can be done in polynomial time, and the algorithm requires a polynomial number of flips, the algorithm terminates in polynomial time and achieves an approximation guarantee of 1/2.