

Approximation Algorithms HW 1

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September 10, 2008

Problem 1.3

Show that S is a vertex cover for G

For S to be a vertex cover of G , every edge in G must have at least one endpoint in S .

After the creation of the depth-first search tree, some edges of G will be in the tree, and some will not.

Because S contains every node except for the leaf nodes, every edge in the tree has at least one of its endpoints in S (edges connecting leaf nodes have one endpoint in S , all other edges have two endpoints in S).

Every edge in G that is not in the tree was not added because it already connected to a node in the tree. The nodes each edge is connected to cannot both be leaf nodes, because if they were, the depth-first search would have traversed further, causing one of the nodes to not be a leaf node. Thus, every one of these edges is connected to at least one non-leaf node, all of which are in S .

Therefore, because every edge in G has at least one endpoint in S , S is a vertex cover for G .

Show that $|S| \leq 2 * OPT$

We are given an algorithm that outputs a vertex cover of size $|S|$. We show that there exists a matching M that is of at least size $\frac{|S|}{2}$. We know that an optimal vertex cover must have size greater than or equal to a matching of the graph. Thus, we know that the optimal vertex cover has size $\frac{|S|}{2}$ or larger, which means our approximation algorithm is off by no more than a factor of 2.

Conjecture: Given graph G , let S be the set of non-leaf vertices of a depth-first search tree of G . Then there exists a matching M for graph G where $|M| \geq \frac{|S|}{2}$.

Proof: Define two matches, M_{even} and M_{odd} . Construct M_{even} by taking all nodes at an *even* depth and matching each with a random child. Construct M_{odd} similarly, and take M to be the larger of M_{even} and M_{odd} . We know this is a valid match because an even node will never match with another even node (nor an odd to an odd), and because every node in S contains at least one

child. Because every node in S is now in either M_{even} or M_{odd} (or both), one of these sets must contain at least $\frac{|S|}{2}$ nodes.

From the proof of Theorem 1.3, we know that $|M| \leq OPT$.

From our conjecture, $\frac{|S|}{2} \leq |M|$

From the previous two equations, $|S| \leq 2 * OPT$