# Homework 3

## Problem 5.1

If we assume that $\mathbf{P} \neq \mathbf{NP}$, and the edge costs do not satisfy the triangle inequality, we can show that the $k$-center problem cannot be approximated within factor $\alpha(n)$ for any poly-time computable function $\alpha(n)$ by showing that such an algorithm could solve the dominating set problem in polynomial time.

We can do this by reducing from dominating set to $k$-center. Assume that there does exist an $\alpha(n)$-approximation for the $k$-center problem where the edge weights do not satisfy the triangle inequality. Let $G = (V, E), k$ be an instance of the dominating set problem. We can then construct a corresponding instance of the $k$-center problem. Let $G' = (V, E')$ be a complete graph with the edge costs given by

$$cost(u,v) = \begin{cases} 1, \text{ if } (u,v) \in E, \\ \alpha(n) + \epsilon, \epsilon > 0 \text{ if } (u,v) \notin E \end{cases}$$

These do not satisfy the triangle inequality, but according to are assumptions above, they do not have to. This construction will also only take polynomial time.

If $dom(G) \leq k$, then $G'$ has a $k$-center of cost 1, and if $dom(G) > k$, then the optimum cost of the $k$-center is $\alpha(n) + \epsilon$.

When run on $G'$, the the $\alpha(n)$-approximation must give a solution of cost 1, because if it gives an answer greater than $\alpha(n)$ when the optimal solution is 1, then it is no longer an $\alpha(n)$-approximation.

Hence, we can distinguish between the two possibilities, solving the dominating set problem in polynomial time. Since, we also know that dominating set is $\mathbf{NP}$-complete, unless $\mathbf{P} = \mathbf{NP}$, this is a contradiction.