Max Flow Min Cut via LP Duality

A path LP for Max Flow. Here is the linear program LP_0 for the max flow problem:

$$\max \sum_{e=(s,u)} f_e - \sum_{e=(u,s)} f_e \quad \text{such that}$$

$$\forall e \quad f_e \le c_e$$

$$\forall u \ne s,t \quad \sum_{e=(v,u)} f_e = \sum_{e=(u,v)} f_e$$

$$\forall e \quad f_e \ge 0$$

I claim that the value of LP_0 is equal to the value of the following linear program LP_1:

$$\max \sum_{p \text{ path from } s \text{ to } t} x_p \quad \text{such that}$$

$$\forall e \quad \sum_{p: e \in p} x_p \le c_e$$

$$\forall p \quad x_p \ge 0$$

To prove one direction of the claim, let $f \ne 0$ be a flow. Let H be a graph that has an edge e whenever $f_e \ne 0$. Then (by flow conservation properties) s is connected to t in H, so there is a path p such that $f_e > 0$ for every $e \in p$. Let x_p be the minimum of f_e for $e \in p$. Let $f'_e = f_e$ if $e \notin p$ and $= f_e - x_e$ if $e \in p$. Then f' is a flow on the graph with capacities $c' - \in \{c_e, c_e - x_p\}$ depending on whether $e \notin p$ or $\in p$. At least one edge disappears: $c_e > 0, c'_e = 0$. By induction on the number of edges we can model flow f' as $\sum x'_q$ with

$$\forall e \quad \sum_{q: e \in q} x'_p \le c_e$$

$$\forall q \quad x'_q \ge 0$$

So $f = \sum x'_q + x_p$ can also be written as a linear combination of flow path variables. Thus, given $(f_e)_e$ feasible for LP_0, we defined a (x_p) feasible for LP_1, with same value, and so Value(LP_0) $\le$ Value(LP_1).

To prove the other direction of the claim, given a linear combination of flow path variables, (x_p), let $f_e = \sum_{p: e \in p} x_p$. It is easy to check that f satisfies the flow constraints. Thus Value(LP_1) $\le$ Value(LP_0). The claim follows.

The dual LP. By the linear programming duality theorem, the value of LP_1 equals the value of its dual, the following linear program LP_2:

$$\min \sum_e y_e c_e \quad \text{such that}$$
$$\forall p \quad \sum_{e: e\in p} y_e \geq 1$$
$$\forall e \quad y_e \geq 0$$

Equivalent integer program. I claim that LP_2 has the same value as the following integer program IP:

$$\min \sum_e y_e c_e \quad \text{such that}$$
$$\forall p \quad \sum_{e: e\in p} y_e \geq 1$$
$$\forall e \quad y_e \in \{0,1\}$$

One direction is obvious: since LP_2 is a relaxation of IP, we have Value(LP_2) $\leq$ Value(IP).

To prove the other direction (sketch), given an optimal solution (y_e) to LP_2, let L denote the set of vertices reachable from s using edges of value $y_e = 0$ only, and $R = V \setminus L$. Note that $s \in L$ and $t \notin L$, so (L, R) is a cut. Let y'_e be 1 if $e \in L \times R$ and 0 otherwise, and let $\alpha = \min\{y_e, e \in L \times R\}$. Let $y"_e = (y_e - \alpha y'_e)/(1-\alpha)$. We have $y = \alpha y' + (1-\alpha) y"$, with $\alpha \in [0,1]$.

Clearly, y' is feasible for LP_2. As to $y"$, clearly $y"_e \geq 0$ for every e. Consider a path p which crosses the cut (L,R) two or more times: write $p = p_1(u,v)p_2$, where (u,v) is the last time that p crosses the cut. Consider the path $p' = p'_1(u,v)p_2$, where p'_1 is a path from s to u that stays entirely in L (it exists by definition of L.) Since p' only crosses the cut once, the constraint for p' is satisfied (by an easy calculation); and that implies $\sum_{e\in p} y_e \geq \sum_{e \in p'} y_e \geq 1$, so the constraint for p is also satisfied. This means that $y"$ is also feasible for LP_2. Thus y, which is optimal, is a convex combination of two feasible solutions: they must both be optimal as well. But y' is an integer solution. So Value(LP_2) $= \sum_e c_e y'_e \geq$ Value(IP). The claim follows.

Relating the integer program to the minimum cut. Finally, I claim that the value of IP is exactly the value of the minimum cut from s to t.

Indeed, given a feasible y, let L be the set of vertices reachable from s using edges such that $y_e = 0$ only and $R = V \setminus L$. Because of the constraints, (L,R) is a cut. Every edge of $L \times R$ has $y_e = 1$, and so the capacity of the cut is $\sum_{e \in L\times R} c_e = \sum_{e\in L \times R} c_e y_e \leq \sum_e c_e y_e$, so Value($IP$) $\geq$ Value(MinCut).

Conversely, given a cut (L,R), let (y_e) be defined by $y_e = 1$ if $e \in L \times R$ and $y_e = 0$ otherwise. Clearly, (y_e) is feasible for IP and its value equals the capacity of the cut, so Value(MinCut) $\geq$ Value(IP). The claim follows.

Concatenating everything:

$$\text{Value(MaxFlow)} = \text{Value}(LP_0) = \text{Value}(LP_1) = \text{Value}(LP_2) = \text{Value}(IP) = \text{Value(MinCut)}.$$