Online auctions analysis

Theorem 1 *The online primal-dual algorithm presented in class is a $(1-1/e-\epsilon)$-approximation for the online auctions problem, where ϵ goes to 0 with $R=\max_{i,j}(b_{ij}/B_i)$.*

$$\max \sum_{ij} y_{ij}b_{ij} \text{ s.t.} \begin{cases} \sum_i y_{ij} & \leq 1 \quad \forall j \\ \sum_i y_{ij}b_{ij} & \leq B_i \quad \forall i \\ y_{ij} & \geq 0 \quad \forall i,j \end{cases} = \min \sum_i B_i x_i + \sum_j z_j \text{ s.t.} \begin{cases} b_{ij}x_i + z_j & \geq b_{ij} \quad \forall (i,j) \\ x_i, z_j & \geq 0 \quad \forall i,j \end{cases}$$

Algorithm upon arrival of j:
$y_{ij} \leftarrow 0$ for all i (y is feasible)
$z_j \leftarrow 0$
increase z_j until (x,z) is feasible (in other words: if there exists i such that $x_i < 1$ then let $z_j \leftarrow \max_i b_{ij}(1-x_i)$.)
if $z_j \neq 0$,
let i be such that the associated constraint is tight ($b_{ij}x_i + z_j = b_{ij}$).
$y_{ij} \leftarrow 1$ and allocate item j to bidder i.
increase $x_i \leftarrow x_i + \Delta$, where Δ is to be determined.

Analysis:

By construction, (x,z) is feasible.

In a given iteration, if we are working on item j and bidder i then the dual objective function increase is b_{ij} and the primal objective function increase is $B_i\Delta + z_j = B_i\Delta + b_{ij} - b_{ij}x_i$. Let $\Delta = \delta \cdot b_{ij}/B_i$. Then primal increase / dual increase is $\delta + 1 - x_i$. Let $\delta = x_i + \eta$ (in other words, the algorithm does $x_i \leftarrow x_i(1+b_{ij}/B_i) + \eta b_{ij}/B_i$). Then primal increase / dual increase is $1+\eta$.

Is (y) feasible? The first constraint is satisfied by construction. As for the second constraint, not quite, but almost: we will prove that *whenever y_{ij} is non-zero, right before the increase we had* $\sum_i y_{ij}b_{ij} \leq B_i$. So in the end we have $\sum_i y_{ij}b_{ij} \leq B_i + \max_j b_{ij} \leq B_i(1+R)$ and in particular $y/(1+R)$ is dual feasible.

With these three properties, we obtain $\text{Value}(y/(1+R)) = \text{Value}(x,z)/(1+\eta)(1+R) \geq \text{OPT}_f/(1+\eta)(1+R)$, and so $\text{Value}(y) \geq \text{OPT}_f/(1+\eta)$. The output brings revenue at least

$$\text{Value}(y) - \sum_{i:i\text{brings revenue } B_i} b_{ij} \geq \text{Value}(y)(1-R)$$

and $\text{OPT}_f \geq \text{OPT}$, so this is a $(1-R)/(1+\eta)$ approximation.

To prove the italicized statement, we will argue that if $\sum_j y_{ij}b_{ij} \geq B_i$ (or in other words, $\sum_{ij} y_{ij}b_{ij}/B_i \geq 1$) then $x_i \geq 1$ and so i will never be chosen again by any future item. Indeed, during one iteration involving i and item k, the increase in $\sum_{ij} y_{ij}b_{ij}/B_i$ is $dt = b_{ik}/B_i$ and the increase in x_i is $dx = (x+\eta)dt$. Since these increases are small, let's go to the continuous world (the rigorous discrete proof is awkward and less intuitive.) When dt increases from 0 to 1, x increases from 0 to some value x_f such that

$$1 = \int_0^1 dt = \int_0^{x_f} \frac{dx}{x+\eta} = \int_\eta^{x_f+\eta} \frac{du}{u} = \ln(\frac{x_f+\eta}{\eta}) = \ln(1 + x_f/\eta).$$

In other words, $x_f = (e-1)\eta$. Setting $\eta = 1/(e-1)$ yields $x_f = 1$, hence the italicized statement.