General-Sum Games: Correlated Equilibria

This lecture introduces a generalization of Nash equilibrium due to Aumann [1] known as correlated equilibrium, which allows for possible dependencies in strategic choices. A daily example of a correlated equilibrium is a traffic light: a red (green) signal suggests that cars should stop (go), and following each suggestion is of course rational.

Following Aumann [2], we present two definitions of correlated equilibrium and we prove their equivalence. In the first, correlated equilibrium is viewed as the natural outcome of Bayesian rationality in information games; in the second, correlated equilibrium is viewed as a natural generalization of Nash equilibrium that allows for correlations in the players' strategic choices.

1 Information Games

Information games are an extension of strategic form games in which the relevant information on which the players base their decisions is modeled explicitly.

1.1 Definition of Information Systems

Definition 1.1 Given a set of individuals $\mathcal{N}$, an *information system* $\mathcal{I}$ is a tuple $(\Omega, (\mathcal{P}_i, \pi_i)_{i \in \mathcal{N}})$, where

- Ω is a nonempty set of states of the world $(\omega \in \Omega)$
- $\mathcal{P}_i$ is an information (or knowledge) partition $(P_i \in \mathcal{P}_i)$
- $\pi_i : \Omega \to [0, 1]$ is a probability measure describing i's beliefs

When an information system is associated with a strategic form game, a state of the world (i.e., an element of Ω) may be an external factor that is exogenous to the structure of the game, or it may be endogenously determined as an outcome of the players' decision-making processes in the game itself. A set of states of the world (i.e., a subset of Ω) is called an *event*.

An element of information partition $\mathcal{P}_i$ at state ω is called an *information set* for player i and is denoted by $P_i(\omega)$. Intuitively, $P_i(\omega)$ is an equivalence class consisting of those states that player i cannot distinguish from ω. Typically, i knows (or is informed of) $P_i(\omega)$ but not ω itself. Based on this knowledge, i can infer *conditional* probabilities over the states in the relevant information set.

Common Prior Assumption Given a set of individuals $\mathcal{N}$ together with an information system $\mathcal{I}$, the *common prior assumption* holds whenever $\pi_i = \pi_j$, for all $i, j \in \mathcal{N}$. This assumption implies that people ascribe different probabilities to uncertain events only because they have access to different information. In other words, in the absence of any differences in information, people would ascribe the same probabilities to events. This feature is one that distinguishes probabilities and utilities: people can have different utilities simply because they have different preferences (*e.g.*, some like coffee, but some like tea).

1.2 Examples of Information Systems

Example 1.2 Consider the following information system, which characterizes two individuals' beliefs about the price of IBM stock tomorrow. The possible states of the world are up and down: *i.e.*, $\Omega = \{U, D\}$. If neither individual can predict the state of the world that will obtain tomorrow, then the information partition of each individual is the trivial partition, namely $\{\Omega\}$. The individuals' beliefs, however, need not satisfy the common prior assumption. For example, the first individual may attribute equal probabilities to both up and down: *i.e.*, $p_1(U) = p_1(D) = \frac{1}{2}$; while the second may attribute probability $\frac{2}{3}$ to up and $\frac{1}{3}$ to down: *i.e.*, $p_2(U) = \frac{2}{3}$ and $p_2(D) = \frac{1}{3}$. The stated probabilities induce conditional probabilities as follows:

$$\begin{array}{rclcrcl} p_1(U|\{U,D\}) & = & \frac{1}{2} & \qquad & p_2(U|\{U,D\}) & = & \frac{2}{3} \\ p_1(D|\{U,D\}) & = & \frac{1}{2} & \qquad & p_2(D|\{U,D\}) & = & \frac{1}{3} \end{array}$$

□

Example 1.3 Now, extending Example 1.2, assume the set of states of the world $\Omega = \{H, M, L\}$. In other words, the price of IBM stock tomorrow is potentially high, medium, or low. If the first individual can predict when the stock price will fall, but cannot distinguish between circumstances in which the price rises or remains the same, this individual's information partition is $\{\{H, M\}, \{L\}\}$. On the other hand, if the second individual can predict when the stock price will rise, but cannot distinguish between circumstances in which the price falls or remains the same, this individual's information partition is $\{\{H\}, \{M, L\}\}$. Assume the individuals ascribe the following common prior probabilities various states of the world: $\pi_i(M) = \frac{1}{2}$, and $\pi_i(H) = \pi_i(L) = \frac{1}{4}$, for $i \in \{1, 2\}$. These probabilities induce conditional probabilities as follows:

$$
\begin{array}{rclcrcl}
p_1(H|\{H,M\}) & = & \frac{1}{3} & \qquad & p_1(H|\{L\}) & = & 0 \\
p_1(M|\{H,M\}) & = & \frac{2}{3} & & p_1(M|\{L\}) & = & 0 \\
p_1(L|\{H,M\}) & = & 0 & & p_1(L|\{L\}) & = & 1
\end{array}
$$

$$
\begin{array}{rclcrcl}
p_2(H|\{H\}) & = & 1 & \qquad & p_2(H|\{M,L\}) & = & 0 \\
p_2(M|\{H\}) & = & 0 & & p_2(M|\{M,L\}) & = & \frac{2}{3} \\
p_2(L|\{H\}) & = & 0 & & p_2(L|\{M,L\}) & = & \frac{1}{3}
\end{array}
$$

□

1.3 Definition and Examples of Information Games

Definition 1.4 An *information game* $\Gamma(\mathcal{I})$ is a strategic form game Γ together with a information system $\mathcal{I}$ for the set of players $\mathcal{N}$, and, for all players $i \in \mathcal{N}$, a strategy $s_i : \Omega \to A_i$: *i.e.*, $\Gamma(\mathcal{I}) = \langle \mathcal{N}, \mathcal{I}, (A_i, s_i, r_i)_{i \in \mathcal{N}} \rangle$.

Definition 1.5 An *adapted strategy* $s_i : \Omega \to A_i$ is a function from the set of states of the world to the set of actions that is "measurable" with respect to the information partition $\mathcal{P}_i$: *i.e.*, $s_i(\omega_1) = s_i(\omega_2)$ whenever $P_i(\omega_1) = P_i(\omega_2)$. A vector of adapted strategies $s = (s_i)_{i \in I}$ is called an *adapted strategy profile.*

The measurability condition imposed on adapted strategies mandates that identical actions be played at indistinguishable states of the world: *i.e.*, strategic decisions can depend only on players' knowledge, encoded in information sets.

Battle of the Sexes (and any strategic form game of complete information) can be viewed as an information game. In doing so, the possible states of the world can be either exogenous or endogenous to the strategic form game. In Example 1.6, the state of the world describes the players' actions, while in Example 1.7, the state of the world is assumed to be independent of the strategic form game.

Example 1.6 Battle of the Sexes can be viewed as an information game with an endogenous set of states of the world containing all possible outcomes of the strategic form game: *i.e.*, $\Omega = \{(B,B),(B,F),(F,B),(F,F)\}$. The woman is conscious of her own actions but is uncertain of the man's. Her knowledge is described by information partition $\mathcal{P}_W = \{\{(B,B),(B,F)\},\{(F,B),(F,F)\}\}$. Similarly, the man's knowledge is described by information partition $\mathcal{P}_M = \{\{(B,B),(F,B)\},\{(B,F),(F,F)\}\}$. Sample (common) prior probabilities are given by $p_W(B,B) = p_W(F,F) = \frac{1}{2}$ and $p_M(B,B) = p_M(F,F) = \frac{1}{2}$. A strategy for the woman might prescribe that she play B on $\{(B,B),(B,F)\}$ and F on $\{(F,B),(F,F)\}$; one for the man might prescribe that he play B on $\{(B,B),(F,B)\}$ and F on $\{(B,F),(F,F)\}$. In Example 2.2, it is argued that these strategies comprise a correlated equilibrium. □

Example 1.7 Battle of the Sexes can also be viewed as an information game by augmenting the strategic form with the following information system, for example:

- $\Omega = \{x, y\}$
- $\mathcal{P}_W = \mathcal{P}_M = \{\{x\}, \{y\}\}$
- $\pi_i(x) = \pi_i(y) = \frac{1}{2}$ for $i \in \{W, M\}$

One strategy profile is given by the following: $s_W(x) = s_M(x) = B$ and $s_W(y) = s_M(y) = F$. In Example 2.3, it is argued that these strategies constitute a correlated equilibrium. □

1.4 Bayesian Rationality

In information games, actions, and therefore payoffs, are random variables, since the state of the world, which is uncertain, dictates actions. Given strategy profile s, the expected payoffs for player i according to player j are computed based on player j's information—specifically, j's beliefs (which are described by probability measure p_j) conditioned on j's knowledge P_j, as follows:

$$\mathbb{E}[r_i(s)|P_j] = \sum_{\omega \in P_j} p_j(\omega|P_j)\, r_i(s(\omega)) \qquad (1)$$

Definition 1.8 Given an information game $\Gamma(\mathcal{I})$, the strategy s_i for player i is *Bayes rational* given strategy s_{-i} iff for all information sets $P_i \in \mathcal{P}_i$ and for all actions $a_i \in A_i$, $\mathbb{E}[r_i(s_i, s_{-i})|P_i] \geq \mathbb{E}[r_i(a_i, s_{-i})|P_i]$.

A strategy s_i is called Bayes rational for player i if it maximizes i's expectation of i's payoffs, given i's beliefs. A player is called Bayes rational who plays Bayes rational strategies. Bayesian rationality is closely tied to correlated equilibrium.

2 Correlated Equilibrium

There are (at least) two equivalent ways to understand correlated equilibrium. Section 2.1 defines the notion in terms of information games. Section 2.2 revisits the concept in strategic form games, where correlated equilibria arise as joint probability distributions (technically, random variables) over action profiles.

2.1 Definition and Example of Correlated Equilibrium

Definition 2.1 Given an information game $\Gamma(\mathcal{I})$ in which the common prior assumption holds, a(n objective) *correlated equilibrium* is an adapted strategy profile s *s.t.* all players are Bayes rational.[2]

Example 2.2 Consider the Battle of the Sexes viewed as an information game as in Example 1.6. Let us abbreviate the woman's information sets as follows: $P_W(B) = \{(B,B),(B,F)\}$ and $P_W(F) = \{(F,B),(F,F)\}$; and similarly, for the man $P_M(B) = \{(B,B),(F,B)\}$ and $P_M(F) = \{(B,F),(F,F)\}$. Now the induced conditional probabilities are as follows:

$$\begin{array}{rclrcl} p_W[(B,B)|P_W(B)] &=& 1 & p_M[(B,B)|P_M(B)] &=& 1 \\ p_W[(B,F)|P_W(B)] &=& 0 & p_M[(F,B)|P_M(B)] &=& 0 \\ p_W[(F,B)|P_W(F)] &=& 0 & p_M[(B,F)|P_M(F)] &=& 0 \\ p_W[(F,F)|P_W(F)] &=& 1 & p_M[(F,F)|P_M(F)] &=& 1 \end{array}$$

The woman is Bayes rational if her adapted strategy s_W prescribes that she is to play B on $\{(B,B),(B,F)\}$ and F on $\{(F,B),(F,F)\}$, and the man is Bayes rational if his adapted strategy s_M prescribes that he is to play B on $\{(B,B),(F,B)\}$ and F on $\{(B,F),(F,F)\}$. The following calculations confirm that these adapted strategies form a correlated equilibrium.

$$\begin{array}{c} \mathbb{E}[r_W(s_W,s_M)|P_W(B)] = 2 \geq 0 = \mathbb{E}[r_W(F,s_M)|P_W(B)] \\ \mathbb{E}[r_W(s_W,s_M)|P_W(F)] = 1 \geq 0 = \mathbb{E}[r_W(B,s_M)|P_W(F)] \\ \mathbb{E}[r_M(s_W,s_M)|P_M(B)] = 1 \geq 0 = \mathbb{E}[r_M(s_W,F)|P_M(B)] \\ \mathbb{E}[r_M(s_W,s_M)|P_M(F)] = 2 \geq 0 = \mathbb{E}[r_M(s_W,B)|P_M(F)] \end{array}$$

□

Example 2.3 Consider the Battle of the Sexes viewed as an information game as in Example 1.7. The induced conditional probabilities are trivial. The woman is Bayes rational if her adapted strategy s_W prescribes that she is to play B on $\{x\}$ and F on $\{y\}$, and the man is Bayes rational if his adapted strategy s_M prescribes that he is to play B on $\{x\}$ and F on $\{y\}$. The following calculations confirm that these adapted strategies form a correlated equilibrium.

$$\begin{array}{c} \mathbb{E}[r_W(s_W,s_M)|\{x\}] = 2 \geq 0 = \mathbb{E}[r_W(F,s_M)|\{x\}] \\ \mathbb{E}[r_W(s_W,s_M)|\{y\}] = 1 \geq 0 = \mathbb{E}[r_W(B,s_M)|\{y\}] \\ \mathbb{E}[r_M(s_W,s_M)|\{x\}] = 1 \geq 0 = \mathbb{E}[r_M(s_W,F)|\{x\}] \\ \mathbb{E}[r_M(s_W,s_M)|\{y\}] = 2 \geq 0 = \mathbb{E}[r_M(s_W,B)|\{y\}] \end{array}$$

□

[2] A *subjective correlated equilibrium* is a correlated equilibrium in an information game in which the common prior assumption does not necessarily hold.

2.2 Correlated Equilibrium Revisited

In addition to the definition of correlated equilibrium in terms of information games presented above, there exists an equivalent formulation of the notion in terms of joint probability distributions (technically, random variables) over action profiles in strategic form games.

Let $\mathcal{A} = \langle U, 2^U, p \rangle$ be a probability space, with nonempty and finite universe U and probability measure p. Define a *correlated strategy* to be a random variable $f : U \to A$ whose values are action profiles. This random variable has underlying probability distribution q, where

$$q[f = a] = \Pr\{f^{-1}(a)\} = \Pr\{\omega \mid f(\omega) = a\} = \sum_{\{\omega \mid f(\omega)=a\}} p(\omega)$$

Player i's expected rewards, given correlated stategy f, are computed as follows:

$$\begin{aligned} \mathbb{E}[r_i(f)] &= \sum_{a \in A} q[f = a] r_i(a) \\ &= \sum_{a \in A} \sum_{\{\omega \mid f(\omega)=a\}} p(\omega) r_i(a) \end{aligned}$$

Player i's conditional expected rewards given action a_i are computed as follows:

$$\begin{aligned} \mathbb{E}[r_i(f) \mid f_i = a_i] &= \sum_{a_{-i} \in A_{-i}} \frac{q[f_{-i} = a_{-i}, f_i = a_i]}{q[f_i = a_i]} r_i(a_i, a_{-i}) \\ &= \sum_{a_{-i} \in A_{-i}} \frac{\sum_{\{\omega \mid f_{-i}(\omega)=a_{-i}, f_i(\omega)=a_i\}} p(\omega)}{\sum_{\{\omega \mid f_i(\omega)=a_i\}} p(\omega)} r_i(a_i, a_{-i}) \end{aligned}$$

Definition 2.4 Given a strategic form game $\Gamma = \langle \mathcal{N}, (A_i, r_i)_{i \in \mathcal{N}} \rangle$, a correlated strategy f is an equilibrium iff for all players $i \in N$ and for all actions $a_i, a_i' \in A_i$,

$$\mathbb{E}[r_i(f_i, f_{-i}) \mid f_i = a_i] \geq \mathbb{E}[r_i(a_i', f_{-i}) \mid f_i = a_i] \tag{2}$$

In other words,

$$\sum_{a_{-i} \in A_{-i}} q[a_{-i} \mid a_i]\, r_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} q[a_{-i} \mid a_i]\, r_i(a_i', a_{-i}) \tag{3}$$

Observation 2.5 *Given a strategic form game $\Gamma = \langle \mathcal{N}, (A_i, r_i)_{i \in \mathcal{N}} \rangle$, the set of correlated equilibria is defined by the following set of linear constraints, and hence a correlated equilibrium can be computed via linear programming:*

$$\sum_{a \in A} q[a]\, r_i(a) \geq \sum_{a \in A} q[a]\, r_i(a_i', a_{-i}), \quad \sum_{a \in A} q[a] = 1, \quad q[a] \geq 0 \tag{4}$$

Example 2.6 The joint probability distribution $q = \left(\frac{1}{2}(B,B), \frac{1}{2}(F,F)\right)$ for Battle of the Sexes forms a correlated equilibrium in the sense of Definition 2.4. The rationality constraints for the woman are satisfied as follows:

$$\begin{array}{rl} & q(B|B)r_W(B,B) + q(F|B)r_W(B,F) \\ = & 2 \\ \geq & 0 \\ = & q(B|B)r_W(F,B) + q(F|B)r_W(F,F) \end{array}$$

$$\begin{array}{rl} & q(B|F)r_W(F,B) + q(F|F)r_W(F,F) \\ = & 1 \\ \geq & 0 \\ = & q(B|F)r_W(B,B) + q(F|F)r_W(B,F) \end{array}$$

Or equivalently, via Observation 2.5

$$\begin{array}{rl} & q(B,B)r_W(B,B) + q(B,F)r_W(B,F) \\ + & q(F,B)r_W(F,B) + q(F,F)r_W(F,F) \\ = & 1 + 0 + 0 + \frac{1}{2} \\ \geq & 1 + 0 + 0 + 0 \\ = & q(B,B)r_W(B,B) + q(B,F)r_W(B,F) \\ + & q(F,B)r_W(B,B) + q(F,F)r_W(B,F) \end{array}$$

$$\begin{array}{rl} & q(B,B)r_W(B,B) + q(B,F)r_W(B,F) \\ + & q(F,B)r_W(F,B) + q(F,F)r_W(F,F) \\ = & 1 + 0 + 0 + \frac{1}{2} \\ \geq & 0 + 0 + 0 + \frac{1}{2} \\ = & q(B,B)r_W(F,B) + q(B,F)r_W(F,F) \\ + & q(F,B)r_W(F,B) + q(F,F)r_W(F,F) \end{array}$$

The constraints are satisfied similarly for the man. □

3 Equivalence Theorem

As noted by Aumann [2], the two stated definitions of correlated equilibrium are equivalent. We prove Aumann's observations here.

Theorem 3.1 *Definitions 2.1 and 2.4 are equivalent; both yield the same notion of correlated equilibrium.*

Proof 3.1 [Definition 2.1 implies Definition 2.4]

Given an information game $\Gamma(\mathcal{I})$, assume the adapted strategy profile s satisfies $\mathbb{E}[r_i(s_i, s_{-i})|P_i] \geq \mathbb{E}[r_i(a_i, s_{-i})|P_i]$, for all players i and all actions $a_i \in A_i$.

We must exhibit a probability space $(U, 2^U, p)$ and a correlated strategy profile $f : U \to A$ such that

$$\mathbb{E}[r_i(f_i, f_{-i}) \mid f_i = a_i] \geq \mathbb{E}[r_i(a_i', f_{-i}) \mid f_i = a_i]$$

for all players i and all actions $a_i, a_i' \in A_i$.

Consider the probability space $(\Omega, 2^\Omega, \pi)$ and the correlated strategy profile s. We show the following: for all players i,

$$\mathbb{E}[r_i(f_i, f_{-i})] \geq \mathbb{E}[r_i(g \circ f_i, f_{-i})]$$

where $g : A_i \to A_i$.

Let $g : A_i \to A_i$.

$$\begin{aligned}
\mathbb{E}[r_i(f_i, f_{-i})] &= \sum_{\omega \in \Omega} \pi(\omega)\, r_i(s(\omega)) \\
&= \sum_{P_i \in \mathcal{P}_i} \Pr[P_i]\, \mathbb{E}[r_i(s_i, s_{-i}) | P_i] \\
&\geq \sum_{P_i \in \mathcal{P}_i} \Pr[P_i]\, \mathbb{E}[r_i(a_i, s_{-i}) | P_i], \quad \text{for all } a_i \in A_i \\
&= \sum_{\omega \in \Omega} \pi(\omega)\, r_i(g(s_i(\omega)), s_{-i}(\omega)), \quad \text{choose } a_i = g(s_i(\omega)) \\
&= \mathbb{E}[r_i(g \circ f_i, f_{-i})]
\end{aligned}$$

The proof relies on the observation that since s_i is an adapted strategy for all players i, f_i and hence $g \circ f_i$ are adapted (correlated) strategies. In particular, $g \circ f_i$ is constant on each information set $P_i \in \mathcal{P}_i$. □

Proof 3.1 [Definition 2.4 implies Definition 2.1]

Given a probability space $(U, 2^U, p)$ and a correlated strategy $f : U \to A$ (i.e., a random variable) such that

$$\mathbb{E}[r_i(f_i, f_{-i}) \mid f_i = a_i] \geq \mathbb{E}[r_i(a_i', f_{-i}) \mid f_i = a_i]$$

for all players i and all actions $a_i' \in A_i$.

We must exhibit an information system $(\Omega, (\mathcal{P}_i, \pi_i)_{i \in \mathcal{N}})$ that comprises part of an information game in which $s : \Omega \to A$ satisfies

$$\mathbb{E}[r_i(s_i, s_{-i}) | P_i] \geq \mathbb{E}[r_i(a_i, s_{-i}) | P_i]$$

for all players i, all information sets $P_i \in \mathcal{P}_i$, and all actions $a_i \in A_i$.

Define $\Omega = U$, $\pi_i = p$ for all players i, and $s = f$. Also, let $\mathcal{P}_i$ be the partition of Ω generated by s: i.e., $\omega, \omega' \in P_i$ iff $s_i(\omega) = s_i(\omega')$.

For all players i, all information sets $P_i \in \mathcal{P}_i$, and all actions $a_i, a_i' \in A_i$,

$$\begin{aligned}
\mathbb{E}[r_i(s_i, s_{-i}) \mid P_i] &= \mathbb{E}[r_i(f_i, f_{-i}) \mid f_i = a_i] \\
&\geq \mathbb{E}[r_i(a_i', f_{-i}) \mid f_i = a_i] \\
&= \mathbb{E}[r_i(a_i', s_{-i}) \mid P_i]
\end{aligned}$$

□

4 Nash Equilibrium Revisited

Nash equilibrium is a special case of correlated equilibrium in which one player's beliefs about the state of the world is independent of any other's. Independent beliefs leads to independent randomizations over the choice of pure strategies.

Definition 4.1 An *independent information system* $\mathcal{I} = (\Omega, (\mathcal{P}_i, \pi_i)_{i \in \mathcal{N}})$ is a special case of an information system where $\pi_i(P_j \cap P_k) = \pi_i(P_j)\pi_i(P_k)$, for all $P_j \in \mathcal{P}_j$, $P_k \in \mathcal{P}_k$, $i, j, k \in \mathcal{N}$.

Definition 4.2 Given an information game $\Gamma(\mathcal{I})$ in which the common prior assumption holds and $\mathcal{I}$ is independent, a(n objective) *Nash equilibrium* is an adapted strategy profile s *s.t.* all players are Bayes rational.

Example 4.3 Consider once again Battle of the Sexes viewed as an information game as in Example 1.6. Recall that the adapted strategy profile presented forms a correlated equilibrium, given beliefs $p_W(B,B) = p_M(B,B) = \frac{1}{2}$ and $p_W(F,F) = p_M(F,F) = \frac{1}{2}$. This is not, however, a Nash equilibrium, as the independence property fails. Abbreviating information sets as in Example 2.2,

$$\begin{aligned}
p_W[P_W(B)]p_W[P_M(B)] &= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \tfrac{1}{4} \neq \tfrac{1}{2} = p_W[(B,B)] \\
p_W[P_W(B)]p_W[P_M(F)] &= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \tfrac{1}{4} \neq 0 = p_W[(B,F)] \\
p_W[P_W(F)]p_W[P_M(B)] &= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \tfrac{1}{4} \neq 0 = p_W[(F,B)] \\
p_W[P_W(F)]p_W[P_M(F)] &= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \tfrac{1}{4} \neq \tfrac{1}{2} = p_W[(F,F)]
\end{aligned}$$

and similarly for the man. On the other hand, both $p_W(B,B) = p_M(B,B) = 1$ and $p_W(F,F) = p_M(F,F) = 1$ satisfy the independence property, and therefore form pure strategy Nash equilibria. Finally, the following probabilities form a mixed strategy Nash equilibrium:

$$\begin{aligned}
p_W(B,B) = p_M(B,B) &= \tfrac{2}{9} & \qquad p_W(B,F) = p_M(B,F) &= \tfrac{4}{9} \\
p_W(F,B) = p_M(F,B) &= \tfrac{1}{9} & \qquad p_W(F,F) = p_M(F,F) &= \tfrac{2}{9}
\end{aligned}$$

since

$$\begin{aligned}
\mathbb{E}[r_W(s_W,s_M)|P_W(B)] = \tfrac{2}{3} \geq \tfrac{2}{3} = \mathbb{E}[r_W(F,s_M)|P_W(B)] \\
\mathbb{E}[r_W(s_W,s_M)|P_W(F)] = \tfrac{2}{3} \geq \tfrac{2}{3} = \mathbb{E}[r_W(B,s_M)|P_W(F)] \\
\mathbb{E}[r_M(s_W,s_M)|P_M(B)] = \tfrac{2}{3} \geq \tfrac{2}{3} = \mathbb{E}[r_M(s_W,F)|P_M(B)] \\
\mathbb{E}[r_M(s_W,s_M)|P_M(F)] = \tfrac{2}{3} \geq \tfrac{2}{3} = \mathbb{E}[r_M(s_W,B)|P_M(F)]
\end{aligned}$$

and

$$\begin{aligned}
p_W[P_W(B)]p_W[P_M(B)] = \left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right) = \tfrac{2}{9} = p_W(B,B) \\
p_W[P_W(B)]p_W[P_M(F)] = \left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right) = \tfrac{4}{9} = p_W(B,F) \\
p_W[P_W(F)]p_W[P_M(B)] = \left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) = \tfrac{1}{9} = p_W(F,B) \\
p_W[P_W(F)]p_W[P_M(F)] = \left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right) = \tfrac{2}{9} = p_W(F,F)
\end{aligned}$$

and similarly for the man. □

References

[1] R. Aumann. Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics*, 1:67–96, 1974.

[2] R. Aumann. Correlated equilibrium as an expression of Bayesian rationality. *Econometrica*, 55(1):1–18, 1987.