An Truthful Approximation Combinatorial Auction

1 Introduction

This document summarizes ideas of an truthful approximately combinatorial auction from the paper [1]. In combinatorial auctions, bidders can express their preferences on a group of items.

A combinatorial auction contains a set **N** of n bidders and a set **G** of k items. A bidder knows his utility function (i.e. his *type*), but keeps it private. The results of an auction is

- an allocation of items among bidders. An allocation is a partial function a from G to P.

$$a : G \rightarrow P \cup \{\text{unallocated}\}$$

- a vector of payments from the bidders to the auctioneer.

Both of which are functions of the bidder's declarations (i.e. *bid*).
Notation: we shall use t to denote a true type, d to denote a bid message/declaration, D to denote a vectors of n declarations, and P for a payment vector.

Assume the Independent Value Model and Quasi-Linear utilities, the utility for a bidder of type t, of bundle $s \subseteq G$ and payment x is

$$u = t(s) - x$$

A *direct mechanism* for combinatorial auctions consists of

- an allocation algorithm f picks an allocation $f(D)$ for each vector of declared type D.
- a payment scheme p that determindes a payment vector $p(D)$ for each vector D.

Let $g_i(D) = [f^{-1}(D)]_i$ denote the bundle obtained by i. If the bidder i has true type t , his utitily from the mechanism is

$$u_i(D) = t(g_i(D)) - p_i(D)$$

where $D = (d_1, d_2, .., d_n)$ is the vector of declaration.

A mechanism $< f, p >$ is *truthful* iff for every bidder i, any vector D of declaration, if D' is the vector obtained from D by replacing the ith coordinate d_i by t, then

$$u_i(D') = t(g_i(D')) - p_i(D') \geq t(g_i(D)) - p_i(D) = u_i(D)$$

In words, a mechanism is truthful if no bidder can be better off by lying , even if other bidders lie.

2 The Generalized Vickrey Auction

The Generalized Vickrey Auctions, also called Vickrey-Clarke-Groves (VCG) mechanism, consists of:

- the allocation that maximizes the sum of the declared valuations of the bidders.

$$f(D) = \text{argmax}_{a \in O} \sum_{i=1}^{n} d_i(a^{-1}(i))$$

  where $a^{-1}(i)$ is the bundle allocated to i by allocation a.

- each bidder receives the sum of declared valuations of all other bidders and pays the auctioneer the sum of such valuation that would have obtained if (s)he had not participated in the auction. That is, each bidder pays the opportunity cost that their presence introduces to all other players.

$$p_j(D) = -\sum_{i=1, i \neq j}^{n} d_i(g_i(D)) + \sum_{i=1, i \neq j}^{n} d_i(g_i(Z))$$

  where $Z_i = D_i$ for any $i \neq j$ and $Z_j(s) = 0$ for any bundle $s \subseteq G$.

Theorem 4.1 The generalized Vickrey auction is a truthful mechanism.

Proof: Consider a bidder $j \in N$ with true type $t \in \Theta$, and any vector D of declaration. Let D' is the vector obtained from D by replacing the ith coordinate d_i by t., i.e, $D'_i = D_i$ for any $i \neq j$ and $D'j = t$.

Since $f(D') = \text{argmax} \sum_{i=1}^{n} d'_i(a^{-1}(i))$, we have

$$\sum_{i=1}^{n} d'_i(g_i(D')) \geq \sum_{i=1}^{n} d'_i(g_i(D))$$

Therefore:

$$\text{LHS} = \sum_{i=1}^{n} d'_i(g_i(D')) = d'_j(g_j(D')) + \sum_{i=1, i \neq j}^{n} d'_i(g_i(D')) = d'_j(g_j(D')) - p_j(D') + \sum_{i=1, i \neq j}^{n} d'_i(g_i(Z))$$

By definition of D', $d'_j(g_j(D)) = t(g_j(D))$ and $d'_i(g_i(D)) = d_i(g_i(D))$ if $i \neq j$

$$\text{LHS} = t(g_j(D')) - p_j(D') + \sum_{i=1, i \neq j}^{n} d_i(g_i(Z))$$

Similarly,

$$\text{RHS} = \sum_{i=1}^{n} d'_i(g_i(D)) = d'_j(g_j(D)) + \sum_{i=1, i \neq j}^{n} d'_i(g_i(D)) = t_j(g_j(D)) + \sum_{i=1, i \neq j}^{n} d_i(g_i(D))$$

$$= t_j(g_j(D)) - p_j(D) + \sum_{i=1, i \neq j}^{n} d_i(g_i(Z))$$

Hence:

$$t(g_j(D')) - p_j(D') + \sum_{i=1, i \neq j}^{n} d_i(g_i(Z)) \geq t_j(g_j(D)) - p_j(D) + \sum_{i=1, i \neq j}^{n} d_i(g_i(Z))$$

$$t(g_j(D')) - p_j(D') \geq t_j(g_j(D)) - p_j(D)$$

3 The Single-Minded Case

Question: *Can we compute the allocation function a in GVA efficiently?*

Bidder i is *single-minded* iff there is a set $s \subseteq G$ of goods and a value $v \in \mathbf{R}$ such that its type t can be described as: $t(s') = v$ if $s \subseteq s'$ and $t(s') = 0$ otherwise.

Proposition 11.5 (paper [2], page 271: The optimal allocation problem in GVA among single-minded biddier is NP-hard.

Proof: Consider a simple case of single-minded bidders where $v_i = 1$ for all $i \in N$. We are going to make a reductin from maximum independent set to this case of optimal allocation.

An *independent set* in a graph $G = (V, E)$ is a set of vertices no two of which are adjacent. A *maximum independent set* is the largest independent set for a given graph.Adding any other node to the maximum independet set forces the set to contain an edge.

We can build a graph representation of an allocation problem as follow:

- Each bidder is a vertex $i \in N$.
- There is an edge (i, j) if $S_i \cap S_j \neq \emptyset$

A set of winners W in the GVA satisfies $S_i \cap S_j = \emptyset$ for any $i, j \in W$ iff the set of vertices corresponding to W is an independent set in the graph.

The optimal allocation in GVA is

$$\text{Max} \sum_{i=1}^{n} v_i = \text{Max}[\sum_{i \in W} \times 1 + \sum_{i \notin W} \times 0] = \text{Max}|W|$$

That is, a maximum independent set of size k exists iff the value of optimal allocation is at least k. This concludes the NP-hardness proof.

4 An Incentive-Compartible Approximation Combinatorial Auction

Question: *If we can solve the optimization problem approximately, can we still maintain the truth revelation properties of GVA?*

4.1 The greedy allocation scheme

- Input: n single-minded bids $b_i =< s_i, v_i >$ for all $i \in N$. Denote $w_i = |s_i|$
- Output: an $\sqrt{k}$-approximate allocation W
- Algorithm:
  1. Sorting the bids in the decresing order of the *average-amount-per-good*: $r_i = \frac{v_i}{\sqrt{w_i}}$
  2. For $i = 1, 2, ..n$, if S_i does not conflict with $\cup_{j \in W} S_j$, then $W \leftarrow W \cup \{i\}$

Theorem 7.2: The greedy allocation scheme with norm $\frac{v_i}{\sqrt{w_i}}$ approximates the optimal allocation within a factor of $\sqrt{k}$.

Proof Let OP denote the optimal allocation, i.e., the set of bids included in the optimal solution. The value of the optimal solution is $\alpha = \sum_{i \in OP} v_i$.

Let GR denote the greedy allocation and its value is $\beta = \sum_{i \in GR} v_i$. We want to show that

$$\alpha \leq \sqrt{k}\beta$$

Without loss of generality, assume that $OP \cap GR = \emptyset$. We have:

$$\beta = \sum_{i \in GR} v_i \geq \sqrt{\sum_{i \in GR} v_i^2} = \sqrt{\sum_{i \in GR} r_i^2 w_i}$$

$$\alpha = \sum_{i \in OP} v_i = \sum_{i \in OP} r_i \sqrt{w_i} \leq \sqrt{\sum_{i \in OP} r_i^2} \sqrt{\sum_{i \in OP} w_i}$$

Because $\sum_{i \in OP} w_i$ is the total number of allocated goods, $\sum_{i \in OP} w_i \leq k$. Hence

$$\alpha \leq \sqrt{\sum_{i \in OP} r_i^2} \sqrt{k}$$

We have to prove that $\sum_{i \in OP} r_i^2 \leq \sum_{i \in GR} r_i^2 w_i$

By assumption $OP \cap GR = \emptyset$, the bids of OP did not enter the greedy solution GR. At any time, any bid i in OP cannot be entered in the partial greedy solution GR. It implies that there is a good $l \in S_i$ that already allocated in the partial greedy solution, i.e., there is a bid j in GR with $r_j \geq r_i$ and $l \in S_j$.

In other words, bid $b_i \in OP$ cannot be granted in the greedy algorithm because there is a bid b_j $(1 \leq j < i)$ in partial greedy solution conflicts with bid b_i, $S_j \cap S_i \neq \emptyset$. There are at most $w_j = |S_j|$ such bids j.

If OP_j is the set of bids of OPT that are associate with b_j, then

$$\sum_{i \in OP_j} r_i^2 \leq r_j^2 w_j$$

Hence:

$$\sum_{i \in OP} r_i^2 = \sum_{j \in GR} \sum_{i \in OP_j} r_i^2 \leq \sum_{j \in GR} r_j^2 w_j$$

4.2 Sufficient Conditions for Truthful Mechanism

Question: *Under what sufficient conditions a mechanism is truthful?*

1. **Exactness**: either $g_j = s$ or $g_j = \emptyset$

   A bidder either gets exactly the set of goods he desires, nothing added, or he get nothing.

2. Monotonicity: if $s' \subseteq s \subset g_j$ and $v' \geq v$, then $s' \subseteq g'_j$

   If the bid $< s, v >$ is granted, then it is also granted when the bidder offer more money for fewer goods.

3. Critical value/payment

   Lemma in a mechanism that satisfies Exactness and Monotonicty, given a bidder j, a set of goods and declarations for all other bidders, there exists a critical value v_c such that:

$$\forall v, v < v_c \Rightarrow g_j = \emptyset$$

$$\forall v, v > v_c \Rightarrow g_j = s$$

   Critical $s \subseteq g_j \Rightarrow p_j = v_c$ A satisfied bidder pays exactly the critical value and still be allocated the good he desires.

4. Participation: $s \nsubseteq g_j \Rightarrow p_j = 0$

   An unsatisfied bidder pay zero.

Theorem 9.6: If a mechanism satisfies Exactness, Monotonicity, Participation, and Critical, then it is a truthful mechansism.

4.3 The greedy payment scheme

Let L be the sorted list obtained in the greedy algorithm. Consider a bid j in L. Let $c(j)$ be the average-amount-per-good of j.

Let $n(j)$ denote the first bid following j in L that has been denied but would have been granted if bid j were not present.

$$n(j) = min\{i | j <_L i, S_j \cap S_i \neq \emptyset, \forall l <_L i, l \neq j, l \text{ granted } \Rightarrow S_l \cap S_i = \emptyset\}$$

For any $j \in N$, *The greedy payment scheme* follows:

$$p_j = \begin{cases} 0 & \text{if } b_j \text{ denined or } n(j) \text{ does not exist} \\ \sqrt{|S_j|} \times c(n(j)) & \text{if } b_j \text{ granted and } n(j) \text{ exists} \end{cases} \tag{1}$$

Theorem 10.2: The mechanism composed of the greedy allocation and payment schemes is truthful for single-minded bidders.
Proof:

1. **Exactness**: The greedy allocation either grants a bid or denies it.

2. **Monotonicity**: Consider 2 bids of bidder $j \in N$, where $s' \subseteq s$ and $v' \geq v$. In sorted order L of $\frac{v}{\sqrt{|s|}}$, $(s', v') >_L (s, v)$. If $< s, v >$ is granted, then $< s', v' >$.

3. **Participation**: satisfied by description in the greedy payment scheme.

4. **Critical**: Assume that bid $n(j)$ exists, if $v_j > \sqrt{|S_j|} \times c(n(j))$ leaves j before $n(j)$ in list L because $\frac{v_j}{\sqrt{|S_j|}} > c(n(j)) = \frac{v_{n(j)}}{\sqrt{|S_{n(j)}|}}$.

   If $v_j < \sqrt{|S_j|} \times c(n(j))$, then $n(j) >_L j$. Hence, bid j is denied because $n(j)$ is granted.

5 Reference

1. Lehmann, O'Callaghan, and Shoham, *Truth Revelation in Approximately Efficient Combinatorial Auctions*, Journal of the ACM, Vol. 49, No. 5, September 2002, pp. 577-602.

2. Blumrosen and Nisan, *Combinatorial Auctions* in book N. Nisan, E. Tardos, T. Roughgarden, and V. Vazirani. Algorithmic Game Theory. Cambridge University Press, 2007.