Zero-Sum Games: Minimax Equilibria

Matching Pennies is a well-known example of a two player, zero-sum game. In this game, each of the players, the *matcher* and the *mismatcher*,[2] flips a coin, and the payoffs are determined as follows. If the coins come up matching (*i.e.*, both heads or both tails), then the matcher wins, so the mismatcher pays the matcher the sum of $1. If the coins do not match (*i.e.*, one head and one tail), then the mismatcher wins, so the matcher pays the mismatcher the sum of $1. In Figure 1, player 1 is the mismatcher and player 2 is the matcher. This game is called zero-sum because the payoffs in each cell of the matrix sum to zero.

	H	T
H	$-1, 1$	$1, -1$
T	$1, -1$	$-1, 1$

Figure 1: Matching Pennies

Rochambeau, or Rock-Paper-Scissors, is another example of a two player, zero-sum game. In this game, each player has three actions, namely rock, paper, or scissors. The payoffs are determined as follows: rock smashes scissors; paper covers rock; but scissors cuts paper. The properties of zero-sum games hold not only in games where payoffs sum to zero, but in any game where the players' interests are diametrically opposed. Figure 2(s) depicts Rock-Paper-Scissors as a zero-sum game; Figure 2(s) depicts Rock-Paper-Scissors as a constant-sum game. The strategic reasoning of rational players is identical across versions.

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

(a) Zero-Sum Game

	R	P	S
R	$1/2, 1/2$	$0, 1$	$1, 0$
P	$1, 0$	$1/2, 1/2$	$0, 1$
S	$0, 1$	$1, 0$	$1/2, 1/2$

(b) Constant-Sum Game

Figure 2: Rock Paper Scissors

In this lecture, we adopt the convention that player 1 is the row player, and her name is Rose; player 2 is the column player, and his name is Colin. Ultimately, Rose is the maximizer and Colin is the minimizer.

[2] The mismatcher is often affectionately referred to as Miss Matcher.

1 Saddle Points: Equilibria in Pure Strategies

Definition A *zero-sum* matrix game is a matrix game with the property that $\sum_{i=1}^{n} R_i(\vec{a}) = 0$, for all action profiles $\vec{a}$.

Example Matching Pennies is an example of a zero-sum matrix game. It consists of a set of players $N = \{1, 2\}$, the matcher and mismatcher respectively, with action sets $A_1 = A_2 = \{H, T\}$, and payoffs:

$$\vec{R}(H, H) = (+1, -1) \qquad \vec{R}(H, T) = (-1, +1)$$
$$\vec{R}(T, H) = (-1, +1) \qquad \vec{R}(T, T) = (+1, -1)$$

Example Zero-sum game matrices are sometimes expressed with only one number in each box, in which case each entry is interpreted as a gain for Rose and a loss for Colin. In matrix notation, $m_{ij} \equiv M(i, j) = R_1(i, j) = -R_2(i, j)$ denotes the amount of money Colin loses and Rose wins (i.e., Colin pays to Rose) in the event the outcome is (i, j), for some $i \in A_1$ and $j \in A_2$.

	L	R
T	1	2
B	4	3

Figure 3: A zero-sum game with a saddle point.

The outcome of the game depicted in Figure 3 is (B, R): the maximizer (Rose) wins 3 and the minimizer (Colin) loses 3. Neither player has any incentive to deviate from this outcome. Rather than play B, Rose could play T, but then she would win only 2; rather than play R, Colin could play L, but then he would lose 4 instead of 3. The entry 3 in this matrix game is called a *saddle point*.

Definition The pair (i_0, j_0) is a *saddle point* iff

$$\min_j M(i_0, j) = M(i_0, j_0) = \max_i M(i, j_0)$$

i.e., it is the minimum entry in its row, and the maximum entry in its column.

If Rose plays row i, she secures a win of at least $\min_j m_{ij}$. Assuming Colin is rational (i.e., a minimizer), Rose maximizes her payoffs by playing an action $i^* \in \arg\max_i \min_j m_{ij}$, which yields the so-called *maximin* value of the game. The action i^* is called a *maximin* action.

If Colin plays column j, he secures a loss of at most $\max_i m_{ij}$. Assuming Rose is also rational (i.e., a maximizer), Colin minimizes his payoffs by playing an action $j^* \in \arg\min_j \max_i m_{ij}$, which yields the so-called *minimax* value. The action j^* is called a *minimax* action.

Proposition In zero-sum matrix games, the minimax value is greater than or equal to the maximin value.

Proof Let $i_0 \in \arg\max_i \min_j m_{ij}$. Analogously, let $j_0 \in \arg\min_j \max_i m_{ij}$. Now there exists j_1 *s.t.* $M(i_0, j_1) = \min_j M(i_0, j)$. Since j_1 is a minimizing response to row i_0, the maximin value of the game $M(i_0, j_1) \leq M(i_0, j_0)$. Similarly, there exists i_1 *s.t.* $M(i_1, j_0) = \max_i M(i, j_0)$. Since i_1 is a maximizing response to column j_0, the minimax value of the game $M(i_1, j_0) \geq M(i_0, j_0)$. Therefore, the minimax value is greater than or equal to the maximin value. □

Corollary Given a zero-sum game, if the minimax and the maximin values of the game coincide, then the game admits a saddle point.

Proof Following our reasoning in the proof of the above proposition, now assume $M(i_0, j_1) = M(i_1, j_0)$. It follows that

$$M(i_0, j_1) = M(i_0, j_0) = M(i_1, j_0)$$

and

$$\min_j M(i_0, j) = M(i_0, j_0) = \max_i M(i, j_0)$$

Therefore, (i_0, j_0) is a saddle point. □

Proposition Given a zero-sum game, if there exists a saddle point (i_0, j_0), then the minimax and the maximin values of the game coincide with the value of the saddle point. Moreover, i_0 is a maximin action and j_0 is a minimax action.

Proof It suffices to show that the maximin value is greater than or equal to the minimax value of the game. Given a zero-sum game, if we assume the existence of a saddle point (i_0, j_0), then

$$\min_j \max_i M(i, j) \leq \max_i M(i, j_0) = M(i_0, j_0) = \min_j M(i_0, j) \leq \max_i \min_j M(i, j)$$

This proof relies on the following facts: for any row i, $\min_j M(i, j) \leq M(i, j_0)$; similarly, for any column j, $\max_i M(i, j) \geq M(i_0, j)$. □

Corollary The minimax and maximin values of a zero-sum game coincide iff the game has a saddle point, in which case the value of the game is precisely the value of the saddle point.

Proof This claim follows immediately from the previous two propositions. □

Proposition Given a zero-sum game, if (i_0, j_0) and (i_1, j_1) are saddle points, then

(a) (i_0, j_1) and (i_1, j_0) are saddle points

(b) $M(i_0, j_0) = M(i_0, j_1) = M(i_1, j_1) = M(i_1, j_0)$

Exercise Prove this proposition.

A saddle point in a zero-sum game is an equilibrium in pure actions. It is an action pair from which neither player has any incentive to deviate. But saddle points need not exist. For example, there are no saddle points in the games of matching pennies or *Rochambeau*. The classic minimax theorem for zero-sum games guarantees the existence of equilibria (i.e., saddle points) in mixed strategies. This result is the focus of the remainder of this lecture.

2 The Minimax Theorem: Equilibria in Mixed Strategies

Recall that an *equilibrium* is an action profile from which no player has any incentive to deviate. In the game of Matching Pennies, there is no equilibrium consisting of pure (*i.e.*, nondeterministic) actions. If row plays H, then the best response of column is T; but if column plays T, the best response of row is not H, but T. Moreover, if row plays T, then the best response of column is H; but if column plays H, then the best response of row is not T, but H. But matching pennies does have a *mixed strategy equilibrium*, namely the probabilistic action profile in which both players choose H with probability $\frac{1}{2}$ and T with probability $\frac{1}{2}$. In fact, Nash and minimax equilibria coincide in zero-sum, two player games.

Theorem If (p^*, q^*) is a Nash equilibrium of a two player, zero-sum game, then

(a) $M(p^*, q^*) = \max_p \min_q M(p, q) = \min_q \max_p M(p, q)$

(b) $p^* \in \arg\max_p \min_q M(p, q)$
$q^* \in \arg\min_q \max_p M(p, q)$

and conversely.

Proof of (a) Observe the following:

$M(p^*, q^*) = \max_p M(p, q^*) \geq \max_p [\min_q M(p, q)] \geq \min_q M(p^*, q) = M(p^*, q^*)$.
$M(p^*, q^*) = \min_q M(p^*, q) \leq \min_q [\max_p M(p, q)] \leq \max_p M(p, q^*) = M(p^*, q^*)$.

Thus, $M(p^*, q^*) = \max_p \min_q M(p, q) = \min_q \max_p M(p, q)$. □

Proof of (b) The proof follows from (a) and the definition of Nash equilibrium: $p^* \in \arg\max_p M(p, q^*) = \arg\max_p \min_q M(p, q)$ and $q^* \in \arg\min_q M(p^*, q) = \arg\min_q \max_p M(p, q)$. □

Proof of Converse Choose maximin strategy $p^* \in \arg\max_p \min_q M(p, q)$ and minimax strategy $q^* \in \arg\min_q \max_p M(p, q)$. By definition, $M(p^*, q^*) = \max_p \min_q M(p, q) = \min_q M(p, q^*)$. In particular, $M(p^*, q^*) \leq M(p^*, q)$, for all q: *i.e.*, q^* is a best-response. Similarly, $M(p^*, q^*) = \min_q \max_p M(p, q) = \max_p M(p, q^*)$; and $M(p^*, q^*) \geq M(p, q^*)$, for all p: *i.e.*, p^* is a best-response. □

3 The Minimax Theorem in Zero-Sum Games & The Duality Theorem in Linear Programming

Historically, the minimax theorem was established by von Neumann (1928) long before Nash's theorem (1951). Indeed, von Neumann (1947) demonstrated the equivalence of the minimax theorem for zero-sum games and the duality theorem of linear programming. In this section, we prove the weak duality theorem; we present sufficient conditions for optimality in linear programming; and we state, but do not prove from first principles, the celebrated strong duality theorem, which identifies necessary conditions for optimality. Instead, we demonstrate the equivalence of this theorem and the minimax theorem for zero-sum games.

We begin with examples of two classic linear programming problems.

Example *The Diet Problem.* Consider a set of m vitamins contained in a set of n foods. The minimum daily requirement of vitamin i is b_i, for all $1 \le i \le m$. The cost (in dollars or calories) per unit of food j is c_j, for all $1 \le j \le n$. The number of units of vitamin i contained in each unit of food j is a_{ij}. How much of each food should be consumed so as to minimize cost without failing to satisfy the minimum daily requirements?

Solution If x_j is the number of units of food j that one consumes daily, then the objective is to minimize $c^T x$. But at the same time, one must satisfy the minimum daily requirement. The term $a_{ij}x_j$ denotes the number of units of vitamin i that are consumed in x_j units of food j. Summing over all foods j yields the constraint $A_i x \ge b_i$, for all vitamins i. Here is the linear program:

$$\begin{aligned} & \min_{x_j} \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \ge b_i, \ \forall i \\ & x_j \ge 0, \ \forall j \end{aligned}$$

Example *The Production Problem.* A company manufactures n products using m resources. The company has in its inventory b_i units of resource i, for all $1 \le i \le m$, and it earns c_j profits by producing one unit of product j, for all $1 \le j \le n$. The number of units of resource i required to produce one unit of product j is a_{ij}. How much of each product should be manufactured so as to maximize total profits, given the stated resource constraints?

Solution If x_j is the number of units of product j that is produced, then the objective is to maximize $c^T x$. But the company's resources are limited. The term $a_{ij}x_j$ denotes the number of units of resource i that is necessary to manufacture x_j units of product j. Summing over all products j yields the constraint $A_i x \le b_i$, for all resources i. Here is the linear program:

$$
\begin{aligned}
&\max_{x_j} \sum_{j=1}^{n} c_j x_j \\
s.t. \quad &\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \ \forall i \\
&x_j \geq 0, \ \forall j
\end{aligned}
$$

Alternative Solution Now consider this problem from the point of view of a graduate student in mechanical engineering. How much is the student willing to pay for the prefabricated resources, given that the production company is not willing to sell its resources at a cost less than the profits it would earn selling postfabricated resources? Here is the linear program for this dual problem:

$$
\begin{aligned}
&\min \sum_{i=1}^{m} b_i y_i \\
s.t. \quad &\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \ \forall j \\
&y_i \geq 0, \ \forall i
\end{aligned}
$$

Definition The general form of the *primal* (P) linear program is as follows:

$$\max_x c^T x \ s.t. \ Ax \leq b, \ x \geq 0 \tag{1}$$

Equivalently, the primal can be expressed in terms of slack variables z, since $\alpha \leq \beta$ iff $\exists \gamma \geq 0 \ s.t. \ \alpha + \gamma = \beta$:

$$\max_{x,z} c^T x \ s.t. \ Ax + z = b, \ x, z \geq 0 \tag{2}$$

In full detail,

$$
\begin{aligned}
\max c_1 x_1 + \ldots + c_n x_n & \\
a_{11} x_1 + \ldots + a_{1n} x_n + z_1 &= b_1 \\
&\vdots \\
a_{m1} x_1 + \ldots + a_{mn} x_n + z_m &= b_m \\
x_1, \ldots, x_n, z_1, \ldots, z_m &\geq 0
\end{aligned}
$$

Definition The corresponding general form of the *dual* (D) linear program is:

$$\min_y y^T b \ s.t. \ A^T y \geq c, \ y \geq 0 \tag{3}$$

Equivalently, the dual can be expressed in terms of slack variables π, since $\alpha \geq \beta$ iff $\exists \gamma \geq 0$ *s.t.* $\alpha - \gamma = \beta$:

$$\min_{y,\pi} y^T b \ s.t. \ A^T y - \pi = c, \ \ y, \pi \geq 0 \tag{4}$$

In full detail,

$$\begin{array}{rcl} \min b_1 y_1 + \ldots + b_m y_m & & \\ a_{11} y_1 + \ldots + a_{1m} y_m - \pi_1 & = & c_1 \\ & \vdots & \\ a_{n1} y_1 + \ldots + a_{nm} y_m - \pi_n & = & c_n \\ y_1, \ldots, y_m, v_1, \ldots, v_n & \geq & 0 \end{array}$$

If the primal problem has n variables and m constraints, then the dual problem has m variables and n constraints. A dual variable y_i is called a *shadow price*, and can be interpreted as the marginal cost of resource i. Each dual constraint j ensures that the marginal cost of all the resources used in the manufacturing of product j is no less than the revenue associated with j. Any slack in the constraints represents the amount that marginal cost exceeds revenue. The objective function minimizes the marginal cost of the resource endowment.

Definition The solution x is *feasible for* P iff $Ax \leq b$ and $x \geq 0$; similarly, the solution y is *feasible for* D iff $A^T y \geq c$ and $y \geq 0$.

Theorem *Weak Duality.* If x is feasible for P and y is feasible for D, then $c^T x \leq y^T b$.

Proof Note that $a^T = a$ if a is a scalar quantity. Thus, $c^T x = (c^T x)^T = x^T c \leq x^T A^T y = (x^T A^T y)^T = y^T A x \leq y^T b$. □

Corollary If x is feasible for P, y is feasible for D, and $c^T x = y^T b$, then x is an optimal solution to P and y is an optimal solution to D.

Proof For all x' feasible for P, $c^T x' \leq y^T b = c^T x$. Thus, x is optimal. For all y' feasible for D, $(y')^T b \geq c^T x = y^T b$. Thus, y is optimal. □

At an optimal solution (in either the primal or the dual), for each inequality at which a slack variable is positive, the value of the corresponding primal or dual variable is zero. This property is called complementary slackness.

Definition The *complementary slackness conditions* can be stated as follows: $u_i \equiv y_i z_i = y_i(b_i - A_i x) = 0, \forall i$ and $v_j \equiv \pi_j x_j = (A_j^T y - c_j) x_j = 0, \forall j$.

Referring back to the production problem, we note that the complementary slackness conditions correspond to market equilibrium conditions.

- If $A_i x < b_i$, for some i, then some of the inventory of resource i is not essential for production, in which case the (optimal) marginal cost, y_i, of resource i is necessarily 0.

- On the other hand, if $y > 0$, then it must be that all of the inventory of resource i (and then some) is necessary for production. Since $A_i x \leq b_i$, it follows that $A_i x = b_i$.

- Similarly, if $A_j^T y > c_j$, for some j, then the cost of producing j exceeds the revenue associated with product j, in which case the optimal value of x_j is necessarily 0.

- On the other hand, if $x_j > 0$, then it must be that the cost of producing product j is no more than the corresponding revenue. Since $A_j^T y \geq c_j$, it follows that $A_j^T y = c_j$.

Theorem If x is feasible for P and y is feasible for D, then x and y satisfy complementary slackness iff $c^T x = y^T b$.

Proof Note that $u_i \geq 0, \forall i$, since x is feasible for P; similarly, $v_j \geq 0, \forall j$, since y is feasible for D. Let $u = \sum_{i=1}^m u_i$ and $v = \sum_{j=1}^n v_j$. Now

$$\begin{aligned}
u + v &= \sum_{i=1}^m u_i + \sum_{j=1}^n v_j \\
&= \sum_{i=1}^m y_i (b_i - A_i x) + \sum_{j=1}^n (A_j^T y - c_j) x_j \\
&= \sum_{i=1}^m y_i b_i - \sum_{i=1}^m y_i A_i x + \sum_{j=1}^n A_j^T y x_j - \sum_{j=1}^n c_j x_j \\
&= \sum_{i=1}^m y_i b_i - \sum_{i=1}^m \sum_{j=1}^n y_i a_{ij} x_j + \sum_{j=1}^n \sum_{i=1}^m y_i a_{ij} x_j - \sum_{j=1}^n c_j x_j \\
&= y^T b - c^T x
\end{aligned}$$

Thus, $u + v = 0$ (*i.e.*, x and y satisfy complementary slackness) iff $y^T b = c^T x$. □

Corollary *Sufficient conditions for optimality.* If x is feasible for P and y is feasible for D, and if x and y satisfy complementary slackness, then x is optimal for P and y is optimal for D.

Theorem *Strong Duality: Necessary conditions for optimality.* If P and D are both feasible, then there exist solutions x and y satisfying complementary slackness, equivalently, $c^T x = b^T y$.

Corollary If P and D are both feasible, then there exists an optimal solution x for P and an optimal solution y for D *s.t.* $c^T x = b^T y$.

We do not prove the strong duality theorem here. Instead, we demonstrate the equivalence of this theorem and the minimax theorem for zero-sum games.

3.1 Duality Theorem ⇒ Minimax Theorem

If Rose plays row i with probability p_i, and Colin plays column j, then Rose's expected payoffs are given by $p^T M_j = \sum_i p_i m_{ij}$. Thus, Rose can secure a win of at least $\min_j p^T M_j$. Assuming Colin is rational, Rose maximizes her payoffs by playing mixed action $p^* \in \arg\max_p \min_j p^T M_j$, which yields the so-called *maximin* value of the game. Thus, the row player's maximization problem is as follows:

$$\max_p \{\min_j p^T M_j\} \; s.t. \; \sum_i p_i = 1, \; p_i \geq 0, \; \forall i \tag{5}$$

Equivalently, this problem can be expressed as the following linear program P_M:

$$\begin{aligned} & \max_p v \\ s.t. \quad & v \leq p^T M_j, \; \forall j \\ & p_i \geq 0, \; \forall i \\ & \sum_i p_i = 1 \end{aligned}$$

Similarly, if Colin plays column j with probability q_j, and Rose plays row i, then Colin's expected payoffs are given by $M_i q = \sum_j m_{ij} q_j$. Thus, Colin can bound his loss by at most $\max_i M_i q$. Assuming Rose is rational, Colin minimizes his payoffs by playing mixed action $q^* \in \arg\min_q \max_i M_i q$, which yields the so-called *minimax* value of the game. Thus, the column player's minimization problem is as follows:

$$\min_q \{\max_i M_i q\} \; s.t. \; \sum_j q_j = 1, \; q_j \geq 0, \; \forall j \tag{6}$$

Equivalently, this problem can be expressed as the following linear program D_M:

$$\begin{aligned} & \min_q v \\ s.t. \quad & v \geq M_i q, \; \forall i \\ & q_j \geq 0, \; \forall j \\ & \sum_j q_j = 1 \end{aligned}$$

The linear programs P_M and D_M are duals of one another, and the celebrated minimax theorem is a corollary of the strong duality theorem.

Minimax Theorem Given a two player, zero-sum game M, there exists p, q *s.t.* p is a feasible solution for P_M and q is a feasible solution for D_M, with common value $v = p^T M q$. In particular, p and q are optimal solutions (i.e., equilibrium strategies).

Observation Given a two player, zero-sum game M, with feasible solutions p and q for P_M and D_M, respectively, the common value $v = p^T Mq$ iff the complementary slackness conditions are satisfied.

Proof Note that $v = p^T Mq = \sum_{i,j} p_i m_{ij} q_j$. Now

$$\begin{aligned}
& \sum_{j=1}^{n} q_j (p^T M_j - v) \\
= & \sum_{j=1}^{n} q_j \left(\sum_{i=1}^{m} p_i m_{ij} - v \right) \\
= & \sum_{i,j} p_i m_{ij} q_j - \sum_{j=1}^{n} q_j v \\
= & \sum_{i,j} p_i m_{ij} q_j - v \\
= & \; 0
\end{aligned}$$

□

Example Two-finger Morra is a two player, zero-sum game like *Rochambeau.* Each player shows either 1 or 2 fingers, and simultaneously guesses how many fingers the other player shows. Thus, each player's action set is of the form (a, b), which reads "show a, guess b." If both players guess correctly or incorrectly, then the game is a draw. Otherwise, if one player guesses correctly and the other player guesses incorrectly, then the player who guesses correctly wins an amount equal to the total number of fingers showing. The payoff matrix for two-finger Morra is depicted in Figure 4.

	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
$(1,1)$	0	2	-3	0
$(1,2)$	-2	0	0	3
$(2,1)$	3	0	0	-4
$(2,2)$	0	-3	4	0

Figure 4: Two-finger Morra

A minimax equilibrium in two-finger Morra is given by $p = q = (0, \frac{3}{5}, \frac{2}{5}, 0)$, with minimax value 0. This solution satisfies the complementary slackness conditions:

$$\begin{aligned}
p_1(M_1 q - v) &= \frac{3}{5}\left(\left\langle 0, -7.5, 3, 0; \frac{3}{5}, 0, 0, \frac{2}{5} \right\rangle - 0 \right) \\
&= \frac{3}{5}(0) \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
p_2(M_2 q - v) &= \frac{3}{5}\left(\left\langle 2,0,0,-3;0,\frac{3}{5},\frac{2}{5},0\right\rangle - 0\right) \\
&= \frac{3}{5}(0) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
p_3(M_3 q - v) &= \frac{2}{5}\left(\left\langle -3,0,0,4;0,\frac{3}{5},\frac{2}{5},0\right\rangle - 0\right) \\
&= \frac{2}{5}(0) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
p_4(M_4 q - v) &= \frac{2}{5}\left(\left\langle 0,3,-4,0;\frac{3}{5},0,0,\frac{2}{5}\right\rangle - 0\right) \\
&= \frac{2}{5}(0) \\
&= 0
\end{aligned}
$$

Similarly, $q_1(p^T M_1 - v) = q_2(p^T M_2 - v) = q_3(p^T M_3 - v) = q_4(p^T M_4 - v) = 0$.

Exercise Write a linear program to compute the value of two-finger Morra. Is the optimal value unique? Are the minimax equilibrium strategies unique?

3.2 Minimax Theorem $\Rightarrow$ Duality Theorem

Given a pair of primal and dual linear programs, we construct a symmetric game with an optimal strategy that gives rise to optimal solutions to the linear programs. In a symmetric game, Rose's payoff, whenever she chooses action i and Colin chooses action j, equals Colin's payoff, whenever he chooses action i and Rose chooses action j, for all actions i, j. *Rochambeau* is an example of a symmetric game; but matching pennies is not.

Definition A two player, zero-sum game is called *symmetric* whenever M is skew-symmetric: i.e., $M = -M^T$.

Definition A two player, zero-sum game is called *fair* whenever $v = 0$.

Theorem Every symmetric, two player, zero-sum game is fair.

Lemma In symmetric, two player, zero-sum games, every maximin strategy p^* for Rose is a minimax strategy for Colin; and every minimax strategy q^* for Colin is a maximin strategy for Rose.

Proof of Theorem Let M be a symmetric, two player, zero-sum game. By the lemma, the maximin strategies p^* equals the minimax strategy q^*. Hence,

$$v = (p^*)^T M q^* = (q^*)^T M^T p^* = -(q^*)^T M p^* = -(p^*)^T M q^* = -v \qquad (7)$$

since $a = a^T$ if a is a scalar quantity. It follows that, $v = 0$. The game is fair. □

Proof of Lemma Let M be a symmetric, two player, zero-sum game. Assume p^* is a maximin strategy for Rose and q^* is a minimax strategy for Colin: i.e.,

$$pM(q^*)^T \leq p^* M(q^*)^T \leq p^* M q^T \qquad (8)$$

for all actions p and q. Computing transposes yields the following:

$$q^* M^T p^T \leq q^* M^T (p^*)^T \leq q M^T (p^*)^T \qquad (9)$$

But now, since M is skew-symmetric,

$$q^*(-M)p^T \leq q^*(-M)(p^*)^T \leq q(-M)(p^*)^T \qquad (10)$$

Equivalently,

$$q^* M p^T \geq q^* M (p^*)^T \geq q M (p^*)^T \qquad (11)$$

Indeed, p^* is a minimax strategy for Colin; q^* is a maximin strategy for Rose. □

Given the pair of primal and dual linear programs in Equations 1 and 3, construct the following skew-symmetric game:

0	A^T	$-c$
$-A$	0	b
c^T	$-b^T$	0

This game is fair, and any strategy that is optimal for Rose is optimal for Colin. Let $w = (x, y, t)^T$ be an optimal strategy in this game, with x n-dimensional, y m-dimensional, and t scalar. Letting J_k denote a k-dimensional column vector of 1s, $w^T J_{m+n+1} = 1$. Moreover, $w^T M w = 0$, since the game is fair.

Theorem Given the pair of primal and dual linear programs in Equations 1 and 3, and the corresponding skew-symmetric game, if $w = (x, y, t)^T$ is an optimal strategy in this game, with $t > 0$, then $x^* = x/t$ and $y^* = y/t$ are the optimal solutions to the primal and dual linear programs, respectively.

Proof Assume $t > 0$. Now

$$\begin{pmatrix} 0 & A^T & -c \\ -A & 0 & b \\ c^T & -b^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} \geq 0$$

Multiplying out yields $A^T y - ct \geq 0$, or equivalently, $A^T y^* \geq c$; $-A^T x + bt \geq 0$, or equivalently, $A^T x^* \leq b$; and $c^T x - b^T y \geq 0$, or equivalently, $c^T x \geq y^T b$. By weak duality, $c^T x \leq y^T b$. Thus, $c^T x = y^T b$, or equivalently, the complementary slackness conditions are satisfied. Finally, since x^* and y^* are feasible solutions, x^* and y^* are indeed optimal solutions for the primal and dual, respectively. □

References

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