

Homework 2

Solution Key

Problem 2.1

Rewrite each of these statements so that negations appear only applied to predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

$$\begin{aligned} (a) \quad & \sim[\forall x\forall yP(x, y)] \\ & = \exists x\exists y\sim P(x, y) \end{aligned}$$

$$\begin{aligned} (b) \quad & \sim[\forall y\exists xP(x, y)] \\ & = \exists y\forall x\sim P(x, y) \end{aligned}$$

$$\begin{aligned} (c) \quad & \sim[\forall y\forall x(P(x, y) \vee Q(x, y))] \\ & = \exists y\exists x\sim(P(x, y) \vee Q(x, y)) \\ & = \exists y\exists x(\sim P(x, y) \wedge \sim Q(x, y)) \end{aligned}$$

$$\begin{aligned} (d) \quad & \sim[(\exists x\exists y\sim P(x, y)) \wedge (\forall x\forall yQ(x, y))] \\ & = (\sim\exists x\exists y\sim P(x, y)) \vee (\sim\forall x\forall yQ(x, y)) \\ & = (\forall x\forall y\sim\sim P(x, y)) \vee (\exists x\exists y\sim Q(x, y)) \\ & = (\forall x\forall yP(x, y)) \vee (\exists x\exists y\sim Q(x, y)) \end{aligned}$$

$$\begin{aligned} (e) \quad & \sim[\forall x(\exists y\forall zP(x, y, z) \wedge \exists z\forall yP(x, y, z))] \\ & = \exists x\sim(\exists y\forall zP(x, y, z) \wedge \exists z\forall yP(x, y, z)) \\ & = \exists x(\sim\exists y\forall zP(x, y, z) \vee \sim\exists z\forall yP(x, y, z)) \\ & = \exists x(\forall y\exists z\sim P(x, y, z) \vee \forall z\exists y\sim P(x, y, z)) \end{aligned}$$

Problem 2.2

Let $S(x, y)$ be the statement “ x can apprehend y ,” where the domain is all people. Use predicates and quantifiers to express the following statements:

Define domains. D = all people. V = VILE agents.

- a. “All V. I. L. E. agents can be apprehended by somebody.”

$$\forall x \in \{\text{VILE agents}\} \exists y, S(y, x)$$

- b. “There is exactly one person whom everybody can apprehend.”

$$\exists x \in D (\forall z S(z, x) \wedge \forall y [y \neq x \implies \exists z \in V, \neg S(z, y)])$$

- c. Express the following statement by two different but equivalent logical expressions using what you know about propositional logic:

“No one can apprehend both Carmen Sandiego and the Contessa.”

$$\forall x \in D (\neg(S(x, \text{CarmenSandiego}) \vee S(x, \text{Contessa})))$$

Using De Morgan’s Law we can rewrite this as

$$\forall x ((\neg S(x, \text{CarmenSandiego})) \wedge (\neg S(x, \text{Contessa})))$$

We can rewrite both statements by pulling a negation outside of the quantified expression:

$$\neg \exists x \neg (\neg(S(x, \text{CarmenSandiego}) \vee S(x, \text{Contessa}))) = \neg \exists x (S(x, \text{CarmenSandiego}) \vee S(x, \text{Contessa}))$$

$$\neg \exists x \in D \neg [(\neg(S(x, \text{CarmenSandiego})) \wedge \neg(S(x, \text{Contessa})))]$$

- d. Now demonstrate that your two statements in part c are logically equivalent.

Problem 2.3

Show that the sum of even integers is even.

Let a_1, a_2, \dots, a_n be even integers. Then for each i , we can write $a_i = 2b_i$ for some integer b_i . Now compute the sum:

$$\begin{aligned} & a_1 + a_2 + \cdots + a_n \\ &= (2b_1) + (2b_2) + \cdots + (2b_n) \\ &= 2(b_1 + b_2 + \cdots + b_n) \end{aligned}$$

Thus $(a_1 + a_2 + \cdots + a_n)$ is even.

Problem 2.4

Given any integers a, b, c , if $a - b$ is odd and $b - c$ is even, what can you say about the parity (odd/even) of $a - c$? Support your answer with a proof.

For $a, b, c \in \mathbb{Z}$, if $a - b$ is odd and $b - c$ is even, then $a - c$ is odd.

Proof. From the definitions of even and odd, $\exists m, n \in \mathbb{Z}$ such that $a - b = 2m + 1$ and $b - c = 2n$.

Then:

$$\begin{aligned} a - c &= (a - b) + (b - c) \\ &= (2m + 1) + 2n \\ &= 2(m + n) + 1 \end{aligned}$$

Let $p = m + n$. The set of integers is closed under addition (the sum of two integers is also an integer), so p is an integer. Therefore $a - c = 2p + 1$ is odd by the definition of odd.

Problem 2.5

Our fancy base 10 number system has an interesting property: a number is divisible by 3 if and only if the sum of its digits is divisible by 3. Show why this is true.

Consider m , an n digit number in base 10.

$$m = d_{n-1}d_{n-2} \dots d_1d_0$$

We can rewrite m as a sum:

$$m = d_0 + 10d_1 + \dots + 10^{n-1}d_{n-1}$$

A little re-arranging yields:

$$m = d_0 + (10 - 1 + 1)d_1 + (10^2 - 1 + 1)d_2 + \dots + (10^{n-1} - 1 + 1)d_{n-1}$$

$$m = d_0 + d_1 + \dots + d_{n-1} + (10 - 1)d_1 + \dots + (10^{n-1} - 1)d_{n-1}$$

$$m = (d_0 + d_1 + \dots + d_{n-1}) + (9d_1 + 99d_2 + \dots + 99 \dots 9d_{n-1})$$

$$m = \left(\sum_{i=0}^{n-1} d_i \right) + 9(d_1 + 11d_2 + \dots + 11 \dots 1d_{n-1})$$

$$m = s + 9x$$

The sum $x = d_1 + 11d_2 + \dots + 11 \dots 1d_{n-1}$ is some positive integer because it is a sum of products of integers. The second term in the sum is therefore $9x = 3 * 3x$, which is clearly divisible by 3.

3|s \Rightarrow **3|m** (if)

Suppose 3 divides s . s can therefore be expressed as $s = 3j, j \in Z$. The last equation above can therefore be written

$$m = 3j + 9x = 3(j + 9x)$$

Since $j + 9x$ is an integer, by the definition of divisibility m must be divisible by 3.

$3|m \Rightarrow 3|s$ (only if)

Suppose 3 divides m . This means that $m = 3k, k \in \mathbb{Z}$. The last equation above can therefore be expressed as

$$3k = s + 9x$$

Rearranging yields this equation

$$s = 3k - 9x = 3(k - 3x)$$

Since $k - 3x$ is an integer, by the definition of divisibility s must be divisible by 3.

$\therefore m$ is divisible by 3 iff s is divisible by 3.

Problem 2.6

Prove or give a counterexample:

- For all integers a, b , and c , if $a|bc$ then $a|b$ or $a|c$.
 - For all primes a and all integers b and c , if $a|bc$ then $a|b$ or $a|c$.
 - For all integers a and b , $a|b$ if and only if $a^2|b^2$.
- This statement is false. Consider the case when $a = 6, b = 2$, and $c = 3$. $a = bc$, so $a|bc$. However, since 6 is greater than either 2 or 3, it is impossible for it to divide either.
 - By the fundamental theorem of arithmetic (Thm 3.3.3), every integer has a unique prime factorization. The integers b, c , and bc can therefore be expressed as follows:

$$b = p_{1b}^{x_{1b}} \cdots p_{mb}^{x_{mb}}, c = p_{1c}^{x_{1c}} \cdots p_{mc}^{x_{mc}}, bc = p_{1b}^{x_{1b}} \cdots p_{mb}^{x_{mb}} p_{1c}^{x_{1c}} \cdots p_{nc}^{x_{nc}}$$

Since each $p_{ij}^{x_{ij}}$ is prime, they are divisible only by 1 and themselves. Their product is divisible by 1 and by any subset of the appropriate set of $p_{ij}^{x_{ij}}$'s. If a divides bc , it must be the case that a is equal to some $p_{ij}^{x_{ij}}$ because a is prime (a is prime $\rightarrow a$ cannot be 1 and cannot be a composite of any larger set of primes). Because $a|bc \Rightarrow a = p_{ij}^{x_{ij}}$, $a|b$ or $a|c$ (b if $j = b$, c if $j = c$).

c. $a|b \Rightarrow a^2|b^2$

By the FTA,

$$a = p_{1a} \cdots p_{ma}, b = p_{1b} \cdots p_{nb}$$

If $a|b$, each p_{ia} must correspond to some p_{jb} . Without loss of generality, we can say that they correspond to the first m p_{jb} 's.

$$b = p_{1a} \cdots p_{ma} p_{(m+1)b} \cdots p_{nb}$$

Then,

$$a^2 = p_{1a}^2 \cdots p_{ma}^2, b^2 = p_{1a}^2 \cdots p_{ma}^2 p_{(m+1)b}^2 \cdots p_{nb}^2$$

$$\frac{b^2}{a^2} = \frac{p_{1a}^2 \cdots p_{ma}^2 p_{(m+1)b}^2 \cdots p_{nb}^2}{p_{1a}^2 \cdots p_{ma}^2} = p_{(m+1)b}^2 \cdots p_{nb}^2$$

Since $p_{(m+1)b}^2 \cdots p_{nb}^2$ is a product of integers, $\frac{b^2}{a^2}$ is an integer and therefore $a^2|b^2$.

$$a^2|b^2 \Rightarrow a|b$$

$$a^2 = p_{1a}^2 \cdots p_{ma}^2, b^2 = p_{1b}^2 \cdots p_{nb}^2$$

$$\frac{b^2}{a^2} = \frac{p_{1a}^2 \cdots p_{ma}^2 p_{(m+1)b}^2 \cdots p_{nb}^2}{p_{1a}^2 \cdots p_{ma}^2} = p_{(m+1)b}^2 \cdots p_{nb}^2$$

$$\frac{b}{a} = \sqrt{\frac{b^2}{a^2}} = \sqrt{p_{(m+1)b}^2 \cdots p_{nb}^2} = p_{(m+1)b} \cdots p_{nb}$$

Since $\frac{b}{a}$ is a product of integers, it is also an integer and therefore $a|b$.

The following problems are non-collaborative—discuss them with no one but the professor and the TAs.

Non-collaborative Problem 2.7

Give the contrapositive, converse, and inverse of the following statements.

a. $\forall d \in \mathbb{Z}, \frac{6}{d} \in \mathbb{Z} \rightarrow d = 3.$

(a) Contrapositive: $\forall d \in \mathbb{Z}, d \neq 3 \rightarrow \frac{6}{d} \notin \mathbb{Z}.$

(b) Converse: $\forall d \in \mathbb{Z}, d = 3 \rightarrow \frac{6}{d} \in \mathbb{Z}.$

(c) Inverse: $\forall d \in \mathbb{Z}, \frac{6}{d} \notin \mathbb{Z} \rightarrow d \neq 3.$

- b. $\forall n \in \mathbb{Z}$, if n is prime, then $n = 2$ or n is odd.
- (a) Contrapositive: $\forall n \in \mathbb{Z}$, if $n \neq 2$ and n is even, then n is not prime.
 - (b) Converse: $\forall n \in \mathbb{Z}$, if $n = 2$ or n is odd, then n is prime.
 - (c) Inverse: $\forall n \in \mathbb{Z}$, if n is not prime, then $n \neq 2$ and n is even.
- c. *If the square of an integer is odd, then the integer is odd.*
- (a) Contrapositive: If an integer is even, then the integer's square is even.
 - (b) Converse: If an integer is odd, then the integer's square is odd.
 - (c) Inverse: If the square of an integer is even, then the integer is even.

Non-collaborative Problem 2.8

Reorder the given premises and use contraposition to show that the conclusion follows from the premises.¹

1. *When I work a logic example without grumbling, you may be sure it is one I understand.*
 2. *The arguments in these examples are not arranged in regular order like the ones I am used to.*
 3. *No easy examples make my head ache.*
 4. *I can't understand examples if the arguments are not arranged in regular order like the ones I am used to.*
 5. *I never grumble at an example unless it gives me a headache.*
- \therefore *These examples are not easy.*

Let p = statement 2, "The arguments in these examples are not arranged in regular order like the ones I am used to."

Let q = "I understand an example."

¹Adapted from Lewis Carroll, *Symbolic Logic* (New York: Dover, 1958), p. 123.

Then according to statement 4, “I can’t understand examples if the arguments are not arranged in regular order like the ones I am used to”,
 $p \rightarrow \sim q$.

Let r = “I grumble at an example.”

The contrapositive of statement 1 is “When I don’t understand an example, you may be sure I work the logic example with grumbling,” or $\sim q \rightarrow r$.

Let s = “An example gives me a headache.”

Statement 5 “I never grumble at an example unless it gives me a headache” means $r \rightarrow s$.

Let t = “These examples are easy.”

By statement 3, “No easy examples make my head ache,” $s \rightarrow \sim t$.

Following the implications, we can conclude $\sim t$, “These examples are not easy.”