CSCI 0220 Discrete Structures and Probability B. Raphael

Homework 3

Solution Key

Reading: S. Epp, Sections 3.4 – 3.7; 4.1 – 4.3.

Problem 3.1

Prove the following statement by contradiction:

For all real numbers x and y, if x is irrational and y is rational then $x - y$ is irrational.

Proof by contradiction: Suppose $x - y$ is rational. By the definition of rational number, $x - y = \frac{a}{b}$ and $y = \frac{c}{d}$, in which a, b, c, d are positive integers.

$$\frac{a}{b} = x - \frac{c}{d}$$
$$\frac{a}{b} + \frac{c}{d} = x$$
$$\frac{ad + bc}{bd} = x$$

By the properties of integers, $(ad + bc)$ is a positive integer, as is bd, so $x = \frac{ad+bc}{bd}$ is rational. However, by assumption x is irrational. This is a contradiction, thus proving the statement.

Problem 3.2

a. *Prove that for all integers a, if a^3 is even then a is even.*

b. *Prove that $\sqrt[3]{2}$ is irrational.*

a. Proof by contraposition:
Contrapositive form: $p \Rightarrow \ q$
If a is odd, then a^3 is odd.

$$a \text{ odd} \Rightarrow a = 2k + 1, k \in \mathbb{Z}$$

$$a^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$$

Since k is an integer, $4k^3 + 6k^2 + 3k$ is also an integer. This means that a^3 can be written in the form $a^3 = 2n + 1, n \in \mathbb{Z}$, so a^3 is odd.

b. Proof by contradiction Assume $\sqrt[3]{2}$ is rational. Then,

$$\sqrt[3]{2} = \frac{p}{q}$$

where p and q are relatively prime integers.

$$(\sqrt[3]{2})^3 = 2 = \frac{p^3}{q^3}$$

$$2q^3 = p^3$$

p and q are integers, so p is even because p^3 is even (see part a for proof). p can therfore be written as $p = 2i, i \in \mathbb{Z}$.

$$2q^3 = (2k)^3 = 8k^3$$

$$q^3 = 4k^3 = 2(2k^3)$$

q and k are integers, so q is even because q^3 is even. q can therefore be written as $q = 2j, j \in \mathbb{Z}$. But p and q are required to be relatively prime, and therefore cannot share a factor of 2. We thus have a contradiction, so $\sqrt[3]{2}$ cannot be rational.

Problem 3.3

On the outside rim of a circular disk the integers from 1 through 30 are painted in random order. Show that no matter what this order is, there must be three successive integers whose sum is at least 45.

This is a proof by contradiction.

If there are not three successive integers whose sum is at least 45, then the sum of each set of three successive integers will be less than 45. Let $x_1, \ldots, x_n$ represent a randomly ordered set of integers from 1 to 30.

$$\begin{aligned} x_1 + x_2 + x_3 &< 45 \\ x_2 + x_3 + x_4 &< 45 \\ &\vdots \\ x_{30} + x_1 + x_2 &< 45 \end{aligned}$$

Summing up all of the equations above, we find that

$$\begin{aligned} 3x_1 + 3x_2 + \ldots + 3x_{30} &< 30 \cdot 45 \\ x_1 + x_2 + \ldots + x_{30} &< 10 \cdot 45 \\ \sum_{i=1}^{30} x_i &< 450 \end{aligned}$$

Since the x_i terms are the numbers from 1 to 30, we can rewrite the sum as

$$\sum_{i=1}^{30} i < 450$$

Using the fact that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, we see that

$$\begin{aligned} \frac{30(30+1)}{2} &< 450 \\ 465 &< 450 \end{aligned}$$

This is a contradiction, so there must be three successive integers whose sum is at least 45.

Problem 3.4

Prove the following using induction:

$6 \mid n(n^2+5), \forall n \geq 1$

Proof: By Induction on n. Let $P(k)$ be the predicate that $6 \mid k(k^2+5)$.

Base Case: When $k = 1$, $k(k^2+5) = 1(1^2+5) = 6$ and we know $6 \mid 6$. Therefore $P(1)$ is true.

Inductive Hypothesis: Assume $P(k)$ in order to prove $P(k+1)$. We must show that $6 \mid (k+1)((k+1)^2+5)$, assuming that $6 \mid k(k^2+5)$. A little algebra helps:

$$\begin{aligned} (k+1)((k+1)^2+5) &= (k+1)(k^2+2*k+1+5) \\ &= k^3+2*k^2+6*k+k^2+2*k+6 \\ &= k^3+3*k^2+8*k+6 \\ &= k^3+3*k^2+(5*k+3*k)+6 \\ &= (k^3+5*k)+(3*k^2+3*k)+6 \\ &= k*(k^2+5)+3*k(k+1)+6 \end{aligned}$$

We know by the induction hypothesis that $6 \mid k(k^2+5)$. We also know that 6 divides itself. To show that 6 divides the middle term, we must show that $k(k+1)$ is even, since then we can pull a 2 out of it to combine with the coefficient 3 to produce the desired 6. While this could be a mini-induction proof of its own, we note that since k and $k+1$ are natural numbers one after the other, one being even implies that the other is odd, and vice versa. Therefore, at least one must be even, which makes the product $k(k+1)$ even.

Since 6 divides all the terms of the expression, it must divide the sum of the terms, so we have

$$6 \mid k*(k^2+5)+3*k(k+1)+6$$

which, since $k*(k^2+5)+3*k(k+1)+6 = (k+1)((k+1)^2+5)$ by our algebra above, implies that

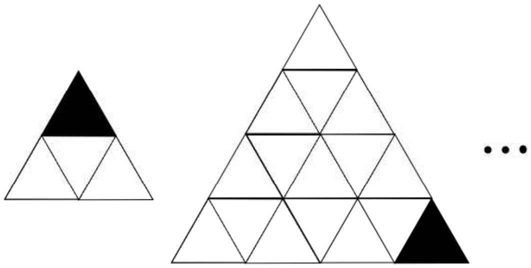
$$6 \mid (k+1)((k+1)^2+5)$$

.

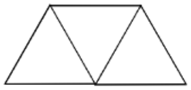
Therefore, we have proved, using mathematical induction, that $6 \mid n(n^2+5)$, $\forall n \geq 1$.

Problem 3.5

Consider the set of equilateral triangles with sides of length 2^n, where $n > 0$, with one 1 x 1 x 1 triangle $\triangle$ missing from any *one of the three corners.*

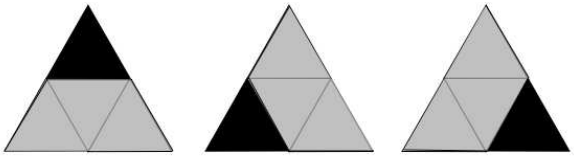


Show that each figure in the set can be covered with tiles composed of three 1 x 1 x 1 triangles △ in the form:



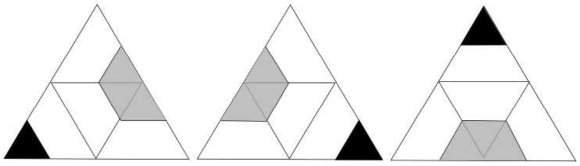
Proof by induction.

We first consider the base case in which $n = 1$, so the triangle has sides of size 2^1. We can show the solution for the three cases in which each of three corners is missing. (You can simply place one shape over the remaining part of the triangle.)



We now assume that we can tile a triangle with sides of length 2^n, and use this assumption to show that we can tile a triangle with sides of length 2^{n+1}.

In each of the three cases in which one of the three corners is missing, we can place a tile along the side of the image to create four triangles with sides of length 2^n with one 1 x 1 x 1 triangle missing. This can be accomplished as follows:



Since each of the four smaller triangles are equilateral triangles with sides of length 2^n with one square "missing," we can use the inductive hypthesis to tile the remaining area.

Problem 3.6

Using induction, prove that

$$\sum_{k=1}^{n+1} k \cdot 2^k = n \cdot 2^{n+2} + 2 \quad \forall n \geq 0$$

Basis Step: $n = 0$

$$\sum_{k=0}^{1} k \cdot 2^k = 0(1) + 1(2) = 2 = 0(2^2) + 2$$

Inductive Step: Assume true for n-1; that is,

$$\sum_{k=0}^{n} k \cdot 2^k = (n-1) \cdot 2^{n+1} + 2$$

We must show that the statement is true for n.

$$\begin{aligned} \sum_{k=0}^{n+1} k \cdot 2^k &= \left(\sum_{k=0}^{n} k \cdot 2^k \right) + (n+1)(2^{n+1}) \\ &= (n-1) \cdot 2^{n+1} + 2 + (n+1)(2^{n+1}) \\ &= 2(n \cdot 2^{n+1}) - 2^{n+1} + 2^{n+1} + 2 \\ &= n \cdot 2^{n+2} + 2 \end{aligned}$$

The following problem is* non-collaborative—*discuss it with no one but the professor and the TAs.

Non-collaborative Problem 3.7

Prove by induction that

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Basis: $n = 1$

$$\sum_{i=1}^{1} i^3 = 1$$

$$\left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1^2 = 1$$

Therefore the statement is true for $n = 1$.

Inductive Step: Assume true for n; that is, assume that

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Show true for $n + 1$

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
&= \frac{(n+1)^2(n^2 + 4(n+1))}{4} = \frac{(n+1)^2(n^2 + 4n + 4)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2 = \left(\frac{(n+1)((n+1)+1)}{2}\right)^2
\end{aligned}$$

Non-collaborative Problem 3.8

The following is a Bonus Problem.

Suppose that n chords are drawn on a circle in such a way that each chord intersects every other, but no three intersect at one point. Prove that the chords divide the circle into $\frac{n^2+n+2}{2}$ *regions. The following example shows the case* $n = 3$*, where the circle is divided into 7 regions.*

Proof by induction. In the 0 case, there is one section of the circle, namely the whole circle.

Suppose that k chords inscribed in the circle form $\frac{k^2+k+2}{2}$ regions in the circle. Then, adding a new chord, $k + 1$, will form an additional $k + 1$

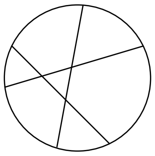


Figure 1: An example where $n = 3$

regions. This is because the chord must intersect k other chords, and thus, by fenceposts, must pass through $k + 1$ regions of the circle, each of which is divided into two by the new chord. Thus:

$$\frac{k^2 + k + 2}{2} + k + 1$$
$$\frac{k^2 + k + 2 + 2k + 2}{2}$$
$$\frac{k^2 + 2k + 1 + k + 1 + 2}{2}$$
$$\frac{(k + 1)^2 + (k + 1) + 2}{2}$$

So, if the statement is true for $k \geq 0$, it is true for $k + 1$.

However, if one does not assume that the chords must intersect within the circle, but may instead intersect on the edge of the circle, the following figure is a counterexample, since it does not have $\frac{k^2+k+2}{2}$, as may be seen by inspection.

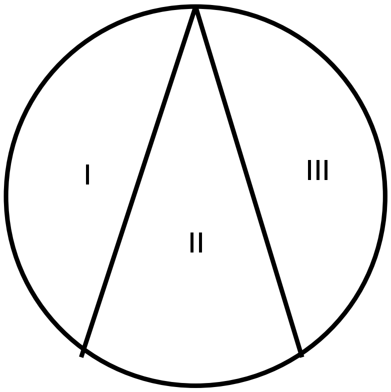


Figure 2: A counterexample