CSCI 0220 **Discrete Structures and Probability** B. Raphael

Homework 9

Solution Key

The following problem is* non-collaborative*—discuss it with no one but the professor and the TAs.

Non-collaborative Problem 9.1

A four of a kind is a five-card poker hand with four of the cards being the same value and the last being any other card. How many ways are there to get four of a kind with a 54-card deck that contains the normal card set plus two jokers? Remember to count the cases where you have four of a kind and a joker.

a. Ways to form four of a kind without a wildcard being one of the four $A = \binom{13}{1}\binom{50}{1}$.

b. Ways to form four of a kind with 1 wild card used as one of the four $B = \binom{13}{1}\binom{4}{3}\binom{2}{1}\binom{48}{1}$. I use 49 here to avoid the double joker with 3 of a kind redundancy, and the last card of that kind.

c. Ways to form four of a kind with both wild cards used as oneo f the four $C = \binom{13}{1}\binom{4}{2}\binom{48}{1}$. Use 48 to avoid the last 2 cards of that kind.

d. Final answer $A + B + C$.

Problem 9.2

Carmen Sandiego is sending an encrypted message back describing each attribute of Wonder Rat, a treacherous villain. Since each attribute has three choices, he represents Wonder Rat's profile as a ternary string. A ternary string is a finite string composed over the alphabet $\{a, b, c\}$. How many ternary strings of length 9 have at least two `a`'s, one `b`, and four `c`'s? Please calculate the numerical result.

Consider a string with two zeroes, one one and four twos. Such a string satisfies the required elements of the string given in the problem, so the remaining three characters in the string can be any number from the set $\{a, b, c\}$.

$$\begin{array}{ccccccccc} a & a & b & c & c & c & c & & \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \end{array}$$

The remaining two slots can be filled as follows:

a	b	c	Permutations	
2	1	6	$\frac{9!}{2!1!6!}$	252
3	1	5	$\frac{9!}{3!1!5!}$	504
4	1	4	$\frac{9!}{4!1!3!}$	630
2	2	5	$\frac{9!}{2!2!5!}$	756
3	2	4	$\frac{9!}{3!2!4!}$	1260
2	3	4	$\frac{9!}{2!3!4!}$	1260

Totalling all of these terms gives us the answer: 4662.

Problem 9.3

How many bit strings of length 10 contain either four consecutive `0`*'s or four consecutive* `1`*'s?*

There are 2^{10} bit strings of length 10. First, let's see how many of those contain four consecutive `0`'s. The sequence of the 4 `0`'s can start at position 1, 2, 3, 4, 5, 6, or 7. In each case, the remaining 6 bits are allowed to be anything, so, as a "first approximation", we can say that there are 7×2^6 different strings of consecutive `0`'s.

However, if we do this, we are *overcounting* since, for example, the string that has its first 8 bits equal to 0 and the last 2 equal to 1 (that is, `0000000011`) will be counted as a valid string *five times* by our method: `0000000011`, `0000000011`, `0000000011`, `0000000011`, and `0000000011`. We need to eliminate these duplicates, which occur whenever we are dealing with strings that have more than 4 consecutive `0`'s.

We can do this by calculating the number of strings with 5 or more consecutive `0`'s using the same method as above. This will give us 6×2^5 such strings, and for each sequence of `0`'s, we will be overcounting exactly one time less than we overcounted above: for example, going back to the string of 8 consecutive `0`'s that was counted five times above, we will see that this method will count is exactly four times. The same will hold for all strings of length ≥ 7 because the number of allowed starting positions for a sequence of 7 `0`'s is exactly one less than the number of allowed starting positions for a sequence of 4 consecutive `0`'s. Subtracting this number will eliminate double counting of strings with more than 4 consecutive 0's. Additionally, there are five cases with two instances of 4 consecutive 0's: `0000100001`, `0000110000`, `1000010000`, `0000100000`, `0000010000` that remain double-counted. Therefore, the number of strings containing 4 or more consecutive `0`'s is:

$$7 \times 2^6 - 6 \times 2^5 - 5 = 251$$

By symmetry, there are another 251 strings which contain four consecutive `1`'s. The question was to determine the number of strings that have either 4 `0`'s or 4 `1`'s. So, we are looking for the size of the union of those two sets (the bit strings with 4 `0`'s and 4 `1`'s). The intersection of the 2 sets (i.e., sequences that contain *both* 4 `0`'s and 4 `1`'s contains 26 elements: `0000001111`, `0000011110`, `0000011111`, `0000101111`, `0000111100`, `0000111101`, `0000111110`, `0000111111`, `0011110000`, `0100001111`, `0111100000`, `0111100001`, `0111110000`, `1000001111`, `1000011110`, `1000011111`, `1011110000`, `1100001111`, `1111000000`, `1111000001`, `1111000010`, `1111000011`, `1111010000`, `1111100000`, `1111100001`, `1111110000`. Therefore, by the inclusion-exclusion principle the size of the union of the two sets is $251+251-26 = 476$.

Problem 9.4

To alleviate some of the gloom of the late New England spring, the class decides to play "Secret Santa". Each of the n students in the class writes his/her name of a piece of paper and places this paper into a bowl at the front of class. Each student then draws a name from the bowl, and must buy a present for the named individual. How many ways are there of choosing Secret Santas so that no student buys his/her own present?

Let us rephrase the problem in a more descriptive manner. The act of drawing names from a hat assigns each person in the group to another person in the same group. A bijective mapping from a finite set to itself is known as a *permutation*. The set containing all of the permutations of a set of size n is denoted S_n. It is a simple counting exercise to demonstrate that $|S_n| = n!$. Suppose $\pi \in S_n$. Then in general, $\pi(p_i) = p_j$. If there exists an i such that $\pi(p_i) = p_i$, π is said to have a fixed point at p_i. If a person draws themself, then that is equivalent to the permutation having a fixed point.

Let $D_n = \{\pi \in S_n : \pi(p_i) \neq p_i \ \forall i\}$. D_n is the set of permutations which have no fixed points; this is precisely the set we are interested in! We know the size of this set should be $n! - |S_n - D_n|$, where the set $S_n - D_n$ is simply the set of permutations containing at least one fixed point. However, $S_n - D_n = \bigcup_{i=1}^{n} F_i$, where F_i is the set of permutations with i fixed points. Computing $|F_i|$ is easy; if we have i fixed points, then we have $(n-i)!$ possibilities. But, there are $\binom{n}{i}$ ways of fixing i points. Thus:

$$|F_i| = \binom{n}{i}(n-i)! = \frac{n!}{i!(n-i)!} \cdot (n-i)! = \frac{n!}{i!}$$

Now we simply use the inclusion-exclusion principle to compute $|S_n - D_n|$:

$$|S_n - D_n| = \frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} - \cdots + \frac{n!}{n!} = n! \sum_{i=1}^{n} \frac{(-1)^i}{i!}$$

Therefore:

$$|D_n| = n! - n! \sum_{i=1}^{n} \frac{(-1)^i}{i!} = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

For the budding mathematicians, this is an interesting result, since:

$$\lim_{n \to \infty} \sum_{i=0}^{n} \frac{(-1)^i}{i!} = \mathrm{e}^{-1}$$

So for large values of n, $\frac{\#(D_n)}{\#(S_n)} \approx \frac{1}{\mathrm{e}}$. In other words, the probability of picking a permutation at random with no fixed points approaches $\frac{1}{\mathrm{e}}$.

An interesting alternative approach to solving this problem was to construct a recurrence relation. Using the notation from before, let $I_n = |S_n - D_n|$. I_n represents the number of permutations with fixed points. Now, we can argue that $I_n = nI_{n-1} + (-1)^n$ based on the idea that any bad permutation in S_{n-1} is also a bad permutation in S_n, but there are n ways of generalizing a permutation in S_{n-1} to S_n. The $(-1)^n$ term reflects the need to account for odd or even values of n. We also know that $I_1 = 1$. This is a nonlinear, inhomogeneous recurrence relation (the most difficult to solve). We can see, however, that our solution above satisfies this recurrence relation. We will show this with induction. Consider the case $n = 1$. Then $|S_1 - D_1| = 1$ and $I_1 = 1$. Now assume true for $n = k$. Consider $n = k + 1$:

$$\begin{aligned} I_{k+1} &= (k+1)I_k + (-1)^{k+1} \\ &= (k+1) \cdot k! \sum_{i=1}^{k} \frac{(-1)^i}{i!} + (-1)^{k+1} \\ &= (k+1)! \left(\left[\sum_{i=1}^{k} \frac{(-1)^i}{i!} \right] + \frac{(-1)^{k+1}}{(k+1)!} \right) \\ &= (k+1)! \sum_{i=1}^{k+1} \frac{(-1)^i}{i!} \end{aligned}$$

Now, since $|D_n| = n! - |S_n - D_n| = n! - I_n$, we can simply substitute and simplify the equation to what we had above.

Problem 9.5

How many different ways are there to list the numbers $1, 2, \ldots, 2n$ *such that the even numbers appear in increasing order and the odd numbers appear in decreasing order?*

There are $2n$ slots into which we are putting numbers. Once we choose which slots get even numbers and which slots get odd numbers, the ordering requirement gives us a unique arrangement. We can thus view this problem as choosing n of the slots to get even numbers, which can be done in $\binom{2n}{n}$ ways.

Alternate approach: We have n even numbers and n odd numbers to assign to $2n$ slots. Because of the ordering constraints, we only need to assign parity to each slot. By the Bookkeeper Rule, this can be done in

$$\frac{(2n)!}{n!\, n!} = \binom{2n}{n}$$

ways.

Problem 9.6

How many integer solutions are there to the system

$$x_1 + x_2 + x_3 + x_4 = 40$$
$$1 \leq x_1 \leq 5$$
$$2 \leq x_2 \leq 7$$
$$3 \leq x_3 \leq 9$$
$$5 \leq x_4$$

There are two ways to solve this problem.

The quick way:

We can redifine the variables such that:

$$y_1 = x_1 - 1$$
$$y_2 = x_2 - 2$$
$$y_3 = x_3 - 3$$
$$y_4 = x_4 - 5$$

and:

$$0 \leq y_1 \leq 4$$
$$0 \leq y_2 \leq 5$$
$$0 \leq y_3 \leq 6$$
$$0 \leq y_4$$

$$y_1 + y_2 + y_3 + y_4 = 29$$

Notice that y_4 has no constraints, since given the maximum values for y_1, y_2, and y_3, $y_4 \geq 14$. Thus, we can set y_1, y_2, and y_3 to any allowed value, and y_4 can take on any value necessary so that the sum equals 40.

There are 5 possible values for y_1, 6 possible values for y_2, and 7 possible values for y_3.

Thus, there are $5 * 6 * 7 = 210$ possible solutions.

The second method:

We can represent the four values of x as 4 boxes, filling up with 40 $\times$'s. We need to assign 11 of these $\times$'s in order to fulfill minimum requirements:

$$\times | \times\times | \times\times\times | \times\times\times\times\times$$

Thus, there are 29 $\times$'s left for 4 boxes, where repitition is allowed. We can use the general formula:

$$\binom{r+n-1}{r}$$

where n is the number of boxes and r is the number of items to put in the boxes. This gives us:

$$\binom{29+4-1}{29} = \binom{32}{29}$$

combinations. However, we need to take away cases where we've gone over the maximum. There are three cases where we've gone over the maximum for exactly one box. For box 1, we would assign 6 $\times$'s to the box initially (along with the minimum requirements for the others), leaving 24 $\times$'s the 4 boxes (since we could also get penalties by filling up box 1 with more than 6). For all three boxes with maximum constraints, this comes to:

$$\binom{24+3}{24} + \binom{23+3}{23} + \binom{22+3}{22}$$

However, we've double counted here. We need to take into account situations where we've gone over the maximum on two boxes. For example, we could fill box 1 and 2 with 12 total $\times$'s, along with the minimum 3 for box 3 and 5 for box 4. By extending this logic to all cases of two penalties, this comes to:

$$\binom{18+3}{18} + \binom{17+3}{17} + \binom{16+3}{16}$$

Finally, we need to add back the case where we've gone over the maximum in all 3 boxes with constraints. This comes to:

$$\binom{11+3}{11}$$

The final answer is:

Combinations - (single penalties) + (double penalties) - (triple penalties), or:

$$\binom{32}{29} - \left(\binom{27}{24} + \binom{26}{23} + \binom{25}{22}\right) + \left(\binom{21}{18} + \binom{20}{17} + \binom{19}{16}\right) - \binom{14}{11}$$

This totals 210 possibilities.

Note: you should know both solutions, as problems on the final might not have the freedom that allowed the shortcut of the first solution!

Problem 9.7

ACME Crime Labs convenes every Thursday for Interpol Alerts at a circular table. Each meeting is attended by 5 male detectives (Nick, Allan, Jason, Josh, and Jon) and three women (Lian, Catherine, and Yoonha).

a. *Nick decided that he wants to sit between two women. He removed one of the chairs so now the round table has 7 chairs. The women arrived first and, as before, sat next to each other. Then the guys arrived and occupied the four remaining seats. Finally, Nick arrived and brought an extra chair, which he placed so as to sit between two women. How many possible arrangements of the TAs at the round table are there in this case? ALSO, for the general case, what if we have w women and m men (including Nick)?*

   There are 3! ways in which the women can sit and before Nick arrives, 4! ways the men can sit. Once Nick comes, he can sit in one of 2 different places thus $3! \cdot 4! \cdot 2 = 288$. For the general case $w! \cdot (m-1)! \cdot (w-1)$.

b. *In how many ways can you sit the men and women around the circular table so that no two women sit next to each other?*

   There are two possible configurations in which two women are not sitting together (a group of three men or two groups of two men). Note that these two configurations are rotationally asymmetric, so you do not need to account for rotationally equivalent solutions. For each configuration, we can order the 5 men in 5! ways and the women in 3! ways. Thus, the total number of configurations in which no two women sit next to each other is $2 \cdot 3! \cdot 5! = 1440$.

c. *Lian decided to treat the TAs to chocolate. There are five flavored chocolates (Amaretto, Hazelnut, Almond, Fish, and Orange) and three plain chocolates (White, Dark and Milk). The TAs will have one chocolate each. The men will have the flavored chocolates, while the women will have the plain chocolates. How many ways are there to distribute chocolates to TAs? What if there are w women, m men, $w + m$ distinct chocolates, w of which are meant for women, and the remaining m meant for men? Is there a bijection to the set of arrangements for part (a)?*

The women can choose chocolats in 3! ways and the men can choose them in 5! ways, $3! \cdot 5! = 720$. For the general case we have $w! \cdot m!$.

There is not a bijection.

d. *The next time around, Lian left eight Milk chocolates for the TAs to eat during the grading session. The men arrived first and decided that (1) each man should have at least one chocolate; and (2) all chocolate should be distributed to men. How many ways are there to distribute these chocolates to the TAs, if the men get their way? (Note that all chocolates are the same, the only thing that matters is how many chocolates each TA got to eat.) What if we have m men and $c \geq m$ pieces of chocolate?*

We begin by giving one chocolate to each man, this leaves 3 which we can distribute using a balls and bins mechanism which gives 7 choose $3 = 35$ ways. In general, we have $c - m$ chocolates to distribute to m men, so the stars and bars formula gives $\binom{(c-m)+m-1}{c-m} = \binom{c-1}{c-m}$.