CSCI 0220 **Discrete Structures and Probability** B. Raphael

Midterm Exam

Solution Key

Closed Book. Duration: 80 minutes

Your Name: ______________________________

Login: ______________________________

- *Problems are weighted differently. The entire exam will be graded out of 100.*
- *Please put your answers on the exam pages themselves (you may request additional scratch paper if needed).* **Remember to write your name at the top of every page.**

Problem 1: _____ / 8 [______________]

Problem 2: _____ / 8 [______________]

Problem 3: _____ / 22 [______________]

Problem 4: _____ / 20 [______________]

Problem 5: _____ / 12 [______________]

Problem 6: _____ / 8 [______________]

Problem 7: _____ / 16 [______________]

Problem 8: _____ / 6 [______________]

Bonus: _____ / 15 [______________]

Total Score: _____ / 100

Name:

Problem 1 (8 pts.)

Let $F(x,y)$ be the statement "x can fool y", where is universe of discourse is all people in the world. Use quantifiers to express each of the following statements.

(a) No one can fool himself or herself.

(b) Everyone can be fooled by someone.

(a) $\neg(\exists x, F(x,x))$ or $\forall x, (\neg F(x,x))$

(b) $\forall x \exists y, F(y,x)$

<u>*Name:*</u>

Problem 2 (8 pts.)

Rewrite the following statement so that negations appear only in front of the predicates P and Q.

(a) $\neg(\forall x \exists y (P(x,y) \vee Q(x,y)))$.

(b) $\neg(\forall x \forall y (P(x,y) \rightarrow \exists z Q(x,z)))$.

(a) $\exists x \forall y ((\neg P(x,y)) \wedge (\neg Q(x,y)))$

(b) $\exists x \exists y (P(x,y) \wedge (\forall z (\neg Q(x,z))))$

Name:

Problem 3 (22 pts.)

Prove that for any integer n, n^3 is odd if and only if n is odd.

Claim: If n is odd, then n^3 is odd.
Proof (direct):
If n is odd, then we can rewrite n as $2k+1$, where k is an integer. Substituting $2k+1$ for n, we find $n^3 = (2k+1)^3$. Distributing and then factoring, we see that $(2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$. Let $j = 4k^3 + 6k^2 + 3k$, which is an integer. Then $n^3 = 2j + 1$, where j is an integer. Thus, by definition of odd numbers, n^3 must be odd.

Claim: If n^3 is odd, then n is odd.
Proof (contraposition):
If n is not odd, then n must be even and $n = 2k$, where k is an integer. Substituting for n, this means that $n^3 = (2k)^3$. Some algebra shows that $(2k)^3 = 8k^3 = 2(4k^3)$. Since $4k^3$ is an integer, we can let $j = 4k^3$. We then see that $n^3 = 2j$, where j is an integer, and therefore n^3 must be even and not odd. Since we showed that if n is not odd, then n^3 is not odd, we can conclude that if n^3 is odd, then n is odd by contraposition.

Having proved both directions, we have shown that n^3 is odd if and only if n is odd.

Name:

Problem 4 (20 pts.)

Prove that $2^n > n^2$ for all integers $n > 4$.

Proof by induction:
Base case:
When $n = 5$, $2^{(5)} = 32$ and $(5)^2 = 25$, and $32 > 25$, proving that the inequality holds for $n = 5$.

Inductive step:
Inductive hypothesis: $2^i > i^2$ for some $i > 4$.
Proof: We need to show that the statement holds for $k = i+1$. Substituting $i + 1$ in for k, we see that this means we need to show $2^{(i+1)} > (i + 1)^2$, or $2 \cdot 2^i > i^2 + 2i + 1$. From our inductive hypothesis, we assume that $2^i > i^2$, which multiplying by 2, also means that $2 \cdot 2^i > 2i^2$. We now need to prove that $2i^2 > i^2 + 2i + 1$, or subtracting i^2 from both sides, that $i^2 > 2i + 1$.

Proof that $i^2 > 2i + 1$, for $i > 4$ by induction:
Base case: When $i = 5$, $(5)^2 = 25$ and $2(5) + 1 = 11$, and $25 > 11$, so the claim holds.
Inductive step: We assume the claim holds for $i = j$ and show that it then holds for $i = j + 1$. Substituting $j + 1$ for i, we find that we need to prove $(j + 1)^2 > 2(j + 1) + 1$, or $j^2 + 2j + 1 > 2j + 3$, which simplifies to $j^2 > 2$. This is clearly true for $j > 4$ (base case: $j = 5$, $25 > 2$; inductive step: assume $h^2 > 2$, $(h + 1)^2 > 2$ when $h^2 + 2h + 1 > 2$, which is valid given the assumption). Thus, we have shown that if $i = j$ is valid, then $i = j + 1$ is also valid.
Since we have shown both the base case and then inductive step, we can conclude that $i^2 > 2i + 1$, for $i > 4$.

Knowing that $i^2 > 2i+1$, for $i > 4$, we can now prove the original claim. We know that $2 \cdot i > 2i^2$, and $2i^2 > i^2+2i+1$, so by transitivity, $2 \cdot i > i^2+2i+1$. Thus, the claim hold for $n = i + 1$, assuming it holds for $n = i$.

After proving the base case and the inductive step, we can conclude that $2^n > n^2$, for $n > 4$.

<u>*Name:*</u>

Problem 5 (12 pts.)

Consider the sets:

$$A = \{a, b, c\}$$
$$B = \{w, x, y, z\}$$
$$C = \{z, a, y\}$$

(a) Give thet set $(A \cup B) - C$*.*

(b) Give the set $A \times A$*.*

(c) Give $\mid \mathcal{P}(A \times A) \mid$*.*

(d) How many relations on A *are there? Justify your answer.*

(a) $\{b, c, w, x\}$

(b) $\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

(c) 2^9

(d) 2^9. Since a relation on a set is defined as a subset of the Cartesian product of the set, the number of relations on A is equivalent to the number of subsets of the Cartesian product of A. All possible subsets of the Cartesian product of A is given by the power set of A. Thus, the number of relations on A is the order of the power set of $A \times A$, which we calculated in part (c).

Name:

Problem 6 (8 pts.)

Recall that a partition of a set defines an equivalence relation. List the ordered pairs in the equivalence relations produced by the following partitions of $\{1, 2, 3, 4, 5, 6\}$

(a) $\{1\}, \{2, 3\}, \{4, 5, 6\}$

(b) $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$.

(a) (1,1), (2,2), (2,3), (3,2), (3,3), (4,4), (4,5), (4,6), (5,4), (5,5), (5,6), (6,4), (6,5), (6,6)

(b) (1,1), (2,2), (3,3), (4,4), (5,5), (6,6)

<u>*Name:*</u>

Problem 7 (16 pts.)

Consider the set $A = \{a, b, c, d\}$ and the relation $R = \{(a, a), (a, b), (a, d), (b, d), (c, c), (c, d), (d, d)\}$.

(a) Show that R is not a partial order on A.

(b) Add a single element to R to obtain a new relation R' such that R' is a partial order on A. Give the added element and prove that R' is a partial order.

(c) List all minimal elements of A given R'.

(d) List all topological sorts of A given R'.

(a) In order for R to be a partial order, R must be reflexive, meaning if $x \in A$, then $(x, x) \in R$. However, $(b, b) \notin R$. Thus, R is not a partial order on A.

(b) Add (b, b) to R to obtain R'. To prove that R' is a partial order, we show that it has the following properties:

- Reflexive: For each $x \in A$, $(x, x) \in R'$. Since $A = \{a, b, c, d\}$ and $(a, a), (b, b), (c, c), (d, d) \in R'$, this property holds.
- Antisymmetry: If $(x, y) \in R'$, then $(y, x) \notin R'$. Since $(a, b), (a, d), (b, d), (c, d) \in R'$ and $(b, a), (d, a), (d, b), (d, c) \notin R'$, this property holds.
- Transitivity: If $(x, y) \in R'$ and $(y, z) \in R'$, then $(x, z) \in R'$. Since $(a, b), (b, d) \in R'$ and $(a, d) \in R'$, this property holds.

Since R' satisfies the three properties of a partial order, we can conclude that R' is a partial order.

(c) a and c

(d) $\{a, b, c, d\}$, $\{a, c, b, d\}$, and $\{c, a, b, d\}$

<u>*Name:*</u>

Problem 8 (6 pts.)

Are there a set A and a relation R on A such that R is both an equivalence relation and partial order on A? Either give an example, or show that no such A and R exist.

Hint*: Look at another problem on this test for inspiration.*

Definition: A relation R on a set A is an equivalence relation if and only if a) It is reflexive, symmetric, and transitive. b) It partitions A into equivalence classes.
Definition: A relation R on a set A is a partial order relation if and only if it is reflexive, antisymmetric, and transitive.
Symmetric: $\forall a, b \in \text{A}, a\text{R}b \Rightarrow b\text{R}a$ **Antisymmetric:** $\forall a, b \in \text{A}, a\text{R}b \wedge b\text{R}a \Rightarrow a = b$
Consider the set $\text{A} = a$ and the relation $\text{R} = (a, a)$. This relation is an equivalence relation because it is derived from a partion of A. Alternatively, it is an equivalence relation because it is reflexive, symmetric, and transitive. Reflexive: $\forall a \in \text{A}, (a, a) \in \text{R}$. Since there is only one element (namely, a) in A, it is clear that this is true. Transitive: $\forall a, b, c \in \text{A}, (a, b) \in \text{R} \wedge (b, c) \in \text{R} \Rightarrow (a, c) \in \text{R}$. Since there is only one element in A, if $a\text{R}b$ and $b\text{R}c$ it must be true that $a = b = c$, which means that clearly $a\text{R}c$ by reflexivity. Symmetric: $\forall a, b \in \text{A}, (a, b) \in \text{R} \Rightarrow (b, a) \in \text{A}$. Since A contains only one element, it must be that $a = b$, so again, $a\text{R}b$ and $b\text{R}a$ by reflexivity. R is a partial order relation because, in addition, it is antisymmetric. Antisymmetric: $\forall a, b \in \text{A}, (a, b) \in \text{R} \wedge (b, a) \in \text{R} \Rightarrow a = b$. Because A contains only one element, it must be that $a = b$, so $(a, b) \in \text{R} \wedge (b, a) \in \text{R} \Rightarrow a = b$

Name:

Problem 9 (15 pts.)

Bonus problem

The game of Mini-nim is defined as follows: Some positive number of sticks are placed on the ground. Two players take turns removing one, two, or three sticks. The player to remove the last stick loses. Prove that the second player has a winning strategy if the number of sticks equals $4k + 1$ for some k.

Claim: The second player has a winning strategy if the number of sticks equals $4k + 1$ for some k.

Proof (by induction):
Base case:
When $k = 0$, there are $4(0) + 1 = 1$ stick. Since the first player goes first and there is only one stick, he/she must pick up the last stick, and by the rules of the game, lose. Therefore, the second player has a winning strategy since he/she wins.

Inductive step:
Inductive hypothesis: The second player has a winning strategy if the number of sticks equals $4i + 1$.
We need to show that this implies that the second player has a winning strategy when $k = i + 1$. In this case, the number of sticks are $4(i + 1) + 1 = (4i + 1) + 4$. The second player can always guarantee that by the end of his/her turn, 4 sticks were picked up, leaving $4i + 1$ sticks. If the first player picks up 1 stick, the second player can pick up 3 sticks. If the first player picks up 2 sticks, the second player can pick up 2 sticks. Likewise, if the first player picks up 3 sticks, the second player can pick up 1 stick. Thus, at the beginning of the first player's next turn, the second payer can guarantee that there are $4i + 1$ sticks. Since this is equivalent to starting a new game with $4i + 1$ sticks, we can use the inductive hypthesis to conclude that the second player has a winning strategy for $k = i + 1$.

Having shown the base case and the inductive hypthesis, we conclude by induction that the second player has a winning strategy if the number of sticks is $4k + 1$, for some k.