Midterm Practice

Solution Key

Problem .1

a. *Simplify the following:*

$$k(x, y, z) = (p(x) \wedge q(y)) \rightarrow \sim r(z)$$

b. *Using your solution from part a. rewrite the following statement without $k(x, y, z)$ and with no negated quantifiers.*
Note: *If you answer part a. incorrectly, but use your answer correctly, you will receive full credit for part b.*

$$\sim \forall x \exists y (\sim \exists z (k(x, y, z)))$$

a.
$$k(x, y, z) = \sim p(x) \vee \sim q(y) \vee \sim r(z)$$

b.
$$\exists x \forall y \exists z, \sim p(x) \vee \sim q(y) \vee \sim r(z)$$

Problem .2

Prove that if $4 \mid n - 3$, *then* $8 \mid n^2 - 1$.

Proof: direct
$4 \mid n-3$ implies, by the definition of divide, that $\exists k \in \mathbb{Z}$, $n-3=4k$.
$n-3=4k$
$n-1=4k+2$
$(n-1)(n+1)=(4k+2)(n+1)$ by multiplication of both sides by a factor
If $n-1=4k+2$, then by algebra it follows $n+1=4k+4$.
$n^2-1=(4k+2)(4k+4)$ by substitution
$n^2-1=16k^2+24k+8$
$n^2-1=8(2k^2+3k+1)$ by factoring the polynomial
Define $m \in \mathbb{Z} = 2k^2+3k+1$. m is an integer because integers are closed over polynomials.
$n^2-1=8m$, $m \in \mathbb{Z}$, therefore $8 \mid n^2-1$ by definition of divide.

Problem .3

Prove by induction that n^3-n is divisible by three for all $n>0$.

Proof: by induction
Basis case: $n=1$, $1^3-1=0$. $3 \mid 0$ because $3z=0$ where $z=0, z \in \mathbb{Z}$.
Inductive hypothesis: $\forall k, k>0$ if $3 \mid n^3-n$ for $n=k$ then it is true for $n=k+1$.
$3 \mid n^3-n$ implies that $\exists m \in \mathcal{Z}, k^3-k=3m$.
I must show that $3 \mid (k+1)^3-(k+1)$:
$(k+1)^3-(k+1)$
$=(k^3+3k^2+3k+1)-(k+1)$
$=k^3+3k^2+2k$
$=(k^3-k)+3k^2+3k$
$=3m+3k^2+3k$ by substitution of the inductive hypothesis
$=3(m+k^2+3k)$
Define $p \in \mathbb{Z} = m+k^2+3k$. p is an integer because integers are closed over polynomials and m and k are both integers.
$(k+1)^3-(k+1)=3p$, therefore $3 \mid (k+1)^3-(k+1)$ by the definition of divide. The claim is true.

Problem .4

Define a relation R on the set of all real numbers, $\mathbb{R}$, as follows: $\forall x, y \in \mathbb{R}$,

$$xRy \Leftrightarrow x^2 \leq y^2$$

Is R a partial order relation? Prove or give a counterexample.
R is not a partial order relation. R is a partial order relation if the set $\mathbb{R}$ over R is reflexive, antisymmetric, and transitive.

i. Reflexive. R is reflexive if xRx. $x^2 \leq x^2$ is trivially true.

ii. NOT Antisymmetric. R is antisymmetric if $\forall x, y \in \mathbb{R}, xRy \wedge yRx \implies x = y$. However consider the counterexample $x = 2, y = -2$.
xRy because $2^2 \leq (-2)^2$
and yRx because $(-2)^2 \leq 2^2$
BUT $x \neq y$ because $2 \neq -2$.
Therefore R is not assymetric. R cannot be a partial order relation.

Problem .5

Consider the sets:
$A = \{1, 2, 3, 5, 8, 13\}$
$B = \{2, 3, 5, 7, 11, 13\}$
$C = \{1, 3, 6, 9, 12\}$

a. *Give the set* $(B - A) \cap C^c$

   $\{7, 11\}$

b. *Give* $|\mathscr{P}(A \cap B)|$

   $A \cap B = \{2, 3, 5, 13\}$
   $2^4 = 16$

Problem .6

Suppose that $h_0, h_1, h_2, h_3, \ldots$ is a sequence defined as follows: $h_0 = 1, h_1, = 2, h_2 = 3, h_k = h_{k-1} + h_{k-2} + h_{k-3}$ *for all integers* $k \geq 3$. *Prove that* $h_n \leq 3^n$

for all integers $n \geq 0$. *Proof:* by strong induction
Bases cases: $n = 0$, $h_0 = 1 = 1 \leq 3^0 = 1$.
$n = 1$, $h_1 = 1 + 2 = 3 \leq 3^1 = 3$.
$n = 2$, $h_2 = 1 + 2 + 3 = 6 \leq 3^2 = 9$.
Inductive hypothesis: $\forall k, 0 < n \leq k$ if $h_n \leq 3^n$ then it is true for $n = k + 1$.
$h_{k+1} = h_k + h_{k-1} + h_{k-2}$
$h_{k+1} \leq 3^k + 3^{k-1} + 3^{k-2}$ by inductive hypothesis
Then it follows also that $h_{k+1} \leq 3^k + 3^k + 3^k$ because $3^k \geq 3^{k-1} \forall k > 0$.
$h_{k+1} \leq 3(3^k)$
$h_{k+1} \leq 3^{k+1}$.
I have proven the inductive hypothesis and the claim is true.

Problem .7

Let F be the relation defined on $\mathbb{Z}$ by as follows: For all $m, n \in \mathbb{Z}$, $mFn \iff 4 \mid (m - n)$.

a) Prove that F is an equivalence relation.
b) Describe the equivalence classes of F.

a) F is an equivalence relation because it is reflexive, symmetric, and transitive.

   i. Reflexive. $\forall m \in \mathbb{Z}$, $4 \mid (m - m)$. $4 \mid 0$ because $4p = 0$ when $p = 0$ and $p \in \mathbb{Z}$.

   ii. Symmetric. $\forall m, n \in \mathbb{Z}$, first suppose mFn. $4 \mid (m - n) \implies 4p = m - n$, $p \in \mathbb{Z}$.
   $-4p = n - m$. Define $q = -p, q \in \mathbb{Z}$. Now, $4q = n - m$, so $4 \mid (n - m)$ and nFm.

   iii. Transitive. $\forall m, n, o \in \mathbb{Z}$, first suppose mFn. $4 \mid (m - n) \implies 4p = m - n$, $p \in \mathbb{Z}$. Also nFo. $4 \mid (n - o) \implies 4q = n - o$, $q \in \mathbb{Z}$. $4p + 4q = m - n + (n - o) = m - o$. Define $r \in \mathbb{Z}, r = p + q$. Therefore $4r = m - o$, so mFo.

b) $4 \mid (x - y) \implies 4p = x - y$, $p \in \mathbb{Z}$.
   It follows that $y = x - 4p$. Therefore, the equivalence class for element $x, x \in \mathbb{Z}$ is:
   $[x] = \{y \in \mathbb{Z} \mid \exists p \in \mathbb{Z}, x - y + 4p = 0\}$. (Alternatively, you could have

$+4p$.)
The equivalence classes of x are all integers such that the difference is a multiple of 4.