Homework 6
A Logic Sandwich

Due: 8:00PM on 3/21/08

Problem 6.1

a. Consider a knowledge base containing just two sentences: $P(a)$ and $P(b)$. Does this knowledge base entail $\forall x\ P(x)$? Explain your answer in terms of models. **Solution: Yes. Because a and b are the only two objects in the model, the predicate $P(x)$ holds for each variable. This is the meaning of $\forall$. OR The domain is not well defined.**

b. Write down a logical sentence such that every world in which it is true contains exactly one object. **Solution:** $\exists x \forall y, x = y$

or?

$$\forall x : P(x)$$

$$\forall x : P(x) \Leftrightarrow x = 0$$

Problem 6.2

Represent the following sentences in first-order logic, using a consistent vocabulary (which you must define)[1]:

a. Every person who buys a policy is smart.

b. No person buys an expensive policy.

c. There is an agent who sells policies only to people who are not insured.

d. There is a barber who shaves all men in town who do not shave themselves.

e. A person born in the UK, each of whose parents is a UK citizen or a UK resident, is a UK citizen by birth.

f. A person born outside the UK, one of whose parents is a UK citizen by birth, is a UK citizen by descent.

[1] From Russell and Norvig 8.6, e-k.

g. Politicians can fool some people all of the time, and they can fool all of the people some of the time, but they can't fool all of the people all of the time.

Solution:

Vocab:

$Buys(x, y, z)$: **person x buys y from z**

$Sells(x, y, z)$: **person x sells y to z**

$Shaves(x, y)$: **person x shaves person y**

$Born(x, c)$: **person x born in country c**

$Parent(x, y)$: **x is parent of y**

$Citizen(x, c, r)$: **x is citizen of c for reason r**

$Resident(x, c)$: **x lives in c**

$Fools(x, y, t)$: **person x fools person y at time t**

$Person(x), Agent(x), Policy(x), Insured(x), Expensive(x), Smart(x), Man(x), Barber(x), Politician(x)$: **x is one of these objects.**

- **a.** $\forall x Person(x) \wedge (\exists y, z Policy(y) \wedge Buys(x, y, z)) \Rightarrow Smart(x)$
- **b.** $\forall x, y, z Person(x) \wedge Policy(y) \wedge Expensive(y) \Rightarrow \neg Buys(x, y, z)$
- **c.** $\exists x Agent(x) \wedge \forall y, z Sells(x, y, z) \Rightarrow \neg Insured(z)$
- **d.** $\exists x Barber(x) \wedge \forall y Man(y) \wedge \neg Shaves(y, y) \Rightarrow Shaves(x, y)$
- **e.** $\forall p Born(p, UK) \wedge (\forall x, Parent(x, p) \Rightarrow \exists r Citizen(x, UK, r) \vee Resident(x, UK)) \Rightarrow Citizen(p, UK, Birth)$
- **f.** $\forall p (\neg Born(p, UK) \wedge \exists x Parent(x, p) \vee Citizen(x, UK, Birth)) \Rightarrow Citizen(p, UK, Descent)$
- **g.** $\forall x Politician(x) \Rightarrow (\exists y \forall t Fools(x, y, t)) \wedge (\exists t \forall y Fools(x, y, t)) \wedge \neg(\forall y \forall t Fools(x, y, t))$

Problem 6.3

A finite probability space, $(\Omega, P, 2^{\Omega})$ [2], where $P : 2^{\Omega} \rightarrow [0, 1]$ has the following properites:

a. $P(\emptyset) = 0$

b. $P(\Omega) = 1$

[2] 2^S where S is a set denotes the power set of S, which is the set of all subsets of S.

c. $P(A \cup B) = P(A) + P(B \backslash A)$

Conditional probability is defined as the probability of A given B and written as $P(A|B) := P(A \cap B)/P(B)$.

a. Let $P_B = P(A|B)$. Show that $(B, P_B, 2^B)$ is a probability space.

b. Show that $P(A|B) = P(B|A) \cdot P(A)/P(B)$.

Solution:

Part A. P_B **must satisfy the properties of a probabilitiy space enumerated above.**

$$\begin{aligned} P_B(\emptyset) &= P(\emptyset|B) \\ &= \frac{P(\emptyset \cap B)}{P(B)} \\ &= \frac{P(\emptyset)}{P(B)} \\ &= 0. \end{aligned}$$

$$\begin{aligned} P_B(\Omega) &= P_B(B) \\ &= P(B|B) \\ &= \frac{P(B \cap B)}{P(B)} \\ &= \frac{P(B)}{P(B)} \\ &= 1. \end{aligned}$$

$$\begin{aligned}
P_B(A \cup C) &= \frac{P((A \cup C) \cap B)}{P(B)} \\
&= \frac{P((A \cap B) \cup (C \cap B))}{P(B)} \\
&= \frac{P(A \cap B) + P((C \cap B) \setminus (A \cap B))}{P(B)} \\
&= P_B(A) + \frac{P((C \cap B) \setminus (A \cap B))}{P(B)} \\
&= P_B(A) + \frac{P((C \cap B) \setminus A) \cup ((C \cap B) \setminus B)}{P(B)} \\
&= P_B(A) + \frac{P((C \cap B) \setminus A)}{P(B)} \\
&= P_B(A) + \frac{P((C \setminus A) \cap B)}{P(B)} \\
&= P_B(A) + P_B(C \setminus A).
\end{aligned}$$

Part B. By definition,

$$P(A|B) = \frac{P(AB)}{P(B)}$$

and

$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(AB)}{P(A)}.$$

We therefore have that

$$P(AB) = P(B|A) \cdot P(A)$$

and so

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}.$$

Problem 6.4

Tim's example of using TAs as test subjects may have had some repercussions. Here is what we know in predicate logic:

$$Meet(Adam, Tim) \wedge$$

$$Meet(Alex, Tim) \wedge$$

$$(\ \forall x, y Meet(x, Tim) \Rightarrow MustEatMushroom(x)\ ) \wedge$$

$$(\; MustEatMushroom(x) \Rightarrow HatesTim(x) \;) \wedge$$
$$(\; HatesTim(x) \wedge HatesTim(y) \Rightarrow LikesEachOther(x, y) \;)$$

a. Bring the above to Prenex normal form

Solution:

$$Meet(Adam, Tim) \wedge$$
$$Meet(Alex, Tim) \wedge$$
$$\neg Meet(x, Tim) \vee MustEatMushroom(x) \wedge$$
$$\neg MustEatMushroom(x) \vee HatesTim(x) \wedge$$
$$\neg HatesTim(x) \vee \neg HatesTim(y) \vee LikesEachOther(x, y)$$

b. Show $LikesEachOther(Adam, Alex)$

Solution: 1,3 / [x = Adam] MustEatMushroom(Adam)

2,3 / [x = Alex] MustEatMushroom(Alex)

4 / [x = Adam] HatesTim(Adam)

4 / [x = Alex] HatesTim(Alex)