Homework 3 (collaboration allowed)

Due: 13 Feb 2003

Reading: first half of Chapter 3 of Spivak

1 Intersections and codimension

Let U be a k-dimensional subspace of $\mathbb{R}^n$, and V an s-dimensional subspace of $\mathbb{R}^n$.

From your linear algebra class, you know that $H = U \cap V$ is also a subspace of $\mathbb{R}^n$, with some dimension h.

(a) Describe, in terms of n, k, and s, the range of possible values for h.

The "generic case," i.e., the value of h you would get almost always if you chose U and V at random (with respect to some reasonable probability distribution), is the lower-bound value.

(b) There's a notion called "codimension" that can simplify the expression for h in this lower-bound case: rather than saying that U has dimension k, we can say "U has codimension $n-k$ in $\mathbb{R}^n$," or simply (in contexts where $\mathbb{R}^n$ is understood), "U has codimension $n-k$." For example, a 5-plane in 12-space has codimension 7. Rewrite the "generic case" answer above by expressing the codimension of H in terms of the codimensions of U and V.

2 "Invariance of dimension" for smooth maps

(This is from Spivak, problem 2-37). Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuously differentiable function.

- Show that f is *not* 1-1. *Hint:* Either $D_1 f(x,y) = 0$ everywhere, which is an easy case, or $D_1 f(p,q) \neq 0$ for some point (p,q). Pick an appropriate neighborhood A of (p,q) and define the function

  $$g : A \longrightarrow \mathbb{R} : (x,y) \mapsto (f(x,y), y).$$

  What can you say about g?

- Generalize to a map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^k$, where $k < n$.

3 Good Stuff

We'll show, some time from now, that if $g : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ $(k \leq n)$ is a smooth function, and $0 \in \mathbb{R}^k$ is a *regular value* of g, i.e., for every point $p \in g^{-1}(\{0\}) = \{x : g(x) = 0\}$, the linear transformation $df(p)$ is surjective, then the set $g^{-1}(\{0\})$ is in fact a *manifold*, a particular kind of object on which it's possible to study analysis especially fruitfully.

For now, you're going to use the statement above to prove that a number of things are manifolds. In class, for instance, we showed that $\mathbb{S}^n$ is a manifold; we did so by looking at the function $g : \mathbb{R}^{n+1} \longrightarrow R : x \mapsto \sum_i (x^i)^2$. The matrix $g'(x)$ is just $[2x_1, \ldots, 2x_{n+1}]$, which is nonzero except at the origin. Hence on $\mathbb{S}^n$, which is the preimage of $\{0\}$ under g, this matrix is never zero. Since the $1 \times n$ matrix $g'(x)$ has at least one nonzero entry for any point $x \in \mathbb{S}^n$, it has rank 1, and we can conclude that $\mathbb{S}^n$ is indeed a manifold (whatever that may be).

In the remainder of this problem, we'll show that $O(n)$, the set of $n \times n$ orthogonal matrices, is also a manifold, in a similar way.

a. We've already seen an isomorphism between M_{nn}, the vector space of $n \times n$ matrices, and $(\mathbb{R}^n)^2$. Describe an isomorphism between $S(n)$, the vector space of symmetric $n \times n$ matrices, and R^p for some value of p.

b. Consider the map

$$f : M_{nn} \longrightarrow S(n) : A \mapsto AA^t.$$

Explain why this map (considered as a map from $(\mathbb{R}^n)^2$ to $\mathbb{R}^p$) is smooth.

c. Recall that a matrix D is orthogonal if $DD^t = I$, where I is the identity matrix. Explain why $O(n) = f^{-1}(\{I\})$. (This is a very easy problem!)

d. To complete the proof that $O(n)$ is a manifold, we have to show that I is a regular value, i.e., that for any matrix $A \in O(n)$, the transformation $df(A)$ is surjective onto $S(n)$. Compute

$$df(A)(B)$$

where B is an arbitrary element of M_{nn}. Hint: write out a limit that describes the directional derivative in the direction B. Now simplify.

e. To show that $df(A)$ is surjective, we need to know that for any $C \in S(n)$, there's a matric B in M_{nn} such that $df(A)(B) = C$. Hint: rewrite the equation $df(A)(B) = C$ using your previous answer to express $df(A)(B)$. Now, because C is symmetric, try rewriting C as $\frac{1}{2}C + \frac{1}{2}C^t$. Now try to guess B. If you need to invert A along the way, recall that it's orthogonal, so $AA^t = A^tA = I$.

4 More Good Stuff

Show that $SL(n)$, the set of $n \times n$ matrices of determinant 1, is also a manifold. Hint: consider the determinant function from M_{nn} to $\mathbb{R}$. It's differentiable at a point A of $SL(n)$ because it's a polynomial in the entries of its argument. The derivative $d\det(A)$ is nonzero if *any* directional derivative is nonzero. So try to find a direction B in which $d\det(A)(B) \neq 0$.

5 Curves in 3-space

Consider a pair of functions f and g from $\mathbb{R}^3$ to $\mathbb{R}$. The sets $f^{-1}(0)$ and $g^{-1}(0)$ are surfaces provided that 0 is a regular value of both f and g, as discussed above. In this situation, what can be said about the intersection X of these two preimages? You might hope, from your familiarity with surfaces, that the intersection would be a collection of curves (or empty).

a. Describe conditions on f and g that ensure that the intersection described is in fact, locally homeomorphic to $\mathbb{R}$ (i.e., for each point x of X, there's a continuous map from a neighborhood U of x in $\mathbb{R}^3$ to an open subset of $\mathbb{R}$ whose restriction to $U \cap X$ is 1-1 and invertible with continuous inverse).

b. Show, by constructing an example, that if your condition is *not* enforced, the intersection can be something other than a set of curves or the empty set.

c. By considering that functions $f(x, y, z) = x^2+y^2+z^2-1$ and $g(x, y, z) = x^2 + y^2 + 4z^2 - 1$, that even if your condition does *not* hold, the intersection of the preimages may still be nice.

6 Tangent vectors

In class/homework we saw four definitions of "tangent vectors to $\mathbb{R}^2$ at the origin":

- The familiar vector space $\mathbb{R}^2$. To give this a name, I'll call it $T^1\mathbb{R}^2$, to indicate that it's our first representation of the tangent space. Because it works equally well as a representation of vectors at points *other* than the origin, I'll actually write $T^1_p\mathbb{R}^2$.
- The set $T^2_p\mathbb{R}^2$ of pairs $(p; v)$, where $p = (0, 0)$ and v is an ordinary vector in $\mathbb{R}^2$. Clearly this can be generalized to represent tangent vectors at points p other than the origin. Recall that the definition of addition and scalar multiplication for $T^2_p\mathbb{R}^2$ were $(p; v_1) + (p; v_2) = (p; v_1+v_2)$, and $\lambda(p; v_1) = (p; \lambda v_1)$, i.e., it's just the ordinary operations on $\mathbb{R}^2$, decorated with a p in front.
- The set of derivations, $T^3\mathbb{R}^2$ on the ring of germs of smooth functions at the origin. Note that for this version of the tangent space, we didn't actually say what to do at points other that the origin.
- The set $T^4\mathbb{R}^2$ of equivalence classes of smooth curves $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ such that $\gamma(0) = (0, 0)$, under the equivalence relation

  $$\gamma \sim \mu \text{ iff } \lim_{t\to 0} \frac{\gamma(t) - \mu(t)}{t} = 0.$$

  Again, we did not define this except at the origin.

In class, we exhibited isomorphisms among all of these.

a. Generalize the definition of T^3 to work for points other than the origin, i.e., give a definition of $T^3_p\mathbb{R}^2$, for an arbitrary point p of $\mathbb{R}^2$, that has to do with the "germs of functions at p."

b. Generalize the definition of T^4 in the same way. Be careful defining a vector-space structure on your set of "curves going through p;" be sure that the sum of two such curves is another such curve!

7 Maps between tangent spaces

Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is differentiable (just *how* differentiable and where it's differentiable are up to you to work out). Furthermore, to make things simple, suppose that $f(0) = 0$. We'll be talking about $T_p^i\mathbb{R}^n$ and $T_q^i\mathbb{R}^k$ for $(i = 1, 2, 3, 4)$, but because we're assuming $f(0) = 0$, we'll in fact always be using $p = 0 \in \mathbb{R}^n$ and $q = 0 \in R^k$.

A map like f is said to "induce a map" from $T_0\mathbb{R}^n \longrightarrow T_0\mathbb{R}^k$; the induced map is called f_*. Here's how it's defined for one particular version of the tangent space:

For T^4 — the "smooth curves through the origin" definition – we can take a curve $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ and produce a new curve, called $f_+(\gamma)$, defined as follows:

$$f_+(\gamma) : \mathbb{R} \longrightarrow \mathbb{R}^k : t \mapsto f \circ \gamma(t).$$

This operation clearly takes the vector space of smooth curves through 0 in $\mathbb{R}^n$ to the vector space of (possibly) smooth curves through the origin in $\mathbb{R}^k$. Several questions remain:

a. To show that f_+ actually takes smooth curves to *smooth* curves, show that $f \circ \gamma$ is a smooth curve. How differentiable does f need to be, and where, for this to work?

b. To show that f_+ is actually well-defined on equivalence classes, we need to show that if γ and μ are equivalent curves in $\mathbb{R}^n$, then $f \circ \gamma$ and $f \circ \mu$ are equivalent curves in $\mathbb{R}^k$.

c. Verify that f_+ is actually a linear map between the two vector spaces.

Finally, by defining $f_*([\gamma]) = [f_+(\gamma)]$ (i.e., by taking the equivalence class of the result), we can define a function

$$f_* : T_0^4\mathbb{R}^n \longrightarrow T_0^4\mathbb{R}^k.$$

This is the "map of tangent spaces induced by f."

Of course, if we can define such an induced map on version 4 of the tangent space, we should be able to define it on the others in a way that's consistent with the isomorphisms we wrote down in class. Let's call those $R_{ij} : T_0^i\mathbb{R}^p \longrightarrow T_0^j\mathbb{R}^p$. So what we just defined was something we could call

f_*^4 (the superscript 4 because it's defined on T^4). We'll now define f_*^3, etc., and we'll want the following to be true:

If v is an element of $T^i\mathbb{R}^n$, it should be the case that

$$f_*^j(R_{ij}(v)) = R_{ij}(f_*^i(v)),$$

i.e., that the "induced map for definition i" is "the same as" the induced map for definition j, where "the same as" means corresponds under our chosen isomorphisms.

I'll get you started: the definition of $f_*^1(v)$ is simply $df(0)(v)$.

c. Define f_*^2.

d. Using function composition as I did in the T^4-case above, define f_*^3.

e. Explain why f_*^3 and f_*^4 are "the same" in the sense described above.

f. Draw a commutative diagram illustrating this.

g. (A thought problem, not to be handed in): Verify mentally that the other f_*^i functions all commute with our isomorphisms, as required above, and figure out how they'd be defined if the condition $f(0) = 0$ were removed (i.e., if we wanted an induced map from $T_p\mathbb{R}^n$ to $T_q\mathbb{R}^k$, where $q = f(p)$).