Lecture 24? Notes

24 April 2003

1 The orthogonal group

Last time we discussed Lie groups; this time I want to look at a couple of specific examples. For example, the set $O(n)$ of all $n \times n$ orthogonal matrices (matrices B for which $B^t B = I$) turns out to be a Lie Group. The multiplication operation is matrix-multiply, inversion is matrix-inverse (which is the same as transpose for this special class of matrices). These are clearly smooth, being the restrictions of smooth maps. The only question is "Is $O(n)$ actually a manifold?"

Consider the map

$$g : M_{nn} \longrightarrow M_{nn} : B \mapsto B^t B.$$

This map is evidently smooth, and in an early homework, you computed its derivative and showed that it had rank $n(n+1)/2$ by showing that every symmetric matrix was in its image. The implicit function theorem applies, and shows that $g^{-1}(I)$ is a smooth submanifold of M_{nn} of dimension $n(n-1)/2$.

The subgroup of $O(n)$ that consists of matrices whose determinant is $+1$ (instead of -1, the only other possibility) is called $SO(n)$, the "special" orthogonal group.

Let's quickly look at the tangent space of $SO(n)$ at the identity element I. To do so, we need to find tangent vectors, which are (in one of our many definitions) just the derivatives of curves through the chosen point. So let

$$\gamma : (-1, 1) \longrightarrow SO(n)$$

be a smooth curve with $\gamma(0) = I$. Because it's a curve in $SO(n)$, we know that for each s,

$$\gamma(s)\gamma(s)^t = I.$$

Differentiating this tells us that

$$\gamma'(s)\gamma(s)^t + \gamma(s)\gamma'(s)^t = 0.$$

At $s = 0$, this becomes

$$\begin{aligned} \gamma'(0)\gamma(0)^t + \gamma(0)\gamma'(0)^t &= 0, \text{ i.e.} \\ \gamma'(0)I + I\gamma'(0)^t &= 0, \text{ i.e.} \\ \gamma'(0) + \gamma'(0)^t &= 0 \end{aligned}$$

which is to say that $\gamma'(0)$ is a skew-symmetric matrix. So every tangent vector to $SO(n)$ at I is a skew-symmetric matrix. Since $T_I SO(n) \subset Y$, where Y denotes

the $n \times n$ skew-symmetric matrices, and both are vector spaces of dimension $n(n-1)/2$, they must in fact be equal.

Summary: The tangent space to $SO(n)$ at the identity is precisely the set of $n \times n$ skew-symmetric matrices. (It's traditionally denoted $so(n)$, by the way.)

At some other point, B, of $SO(n)$ the tangent space is also relatively simple: $T_B SO(n)$ is, after all, just $(L_B)_*(T_I SO(n))$, i.e.,

$$T_B SO(n) = B \cdot so(n) = \{BR : R \in so(n)\}.$$

That means that any tangent vector at B is just B times some skew-symmetric matrix. The same argument holds for *right* multiplication, so any tangent vector at B is just some skew-symmetric matrix times B.

2 Application