Exam 1, updated version of problem 3

Due: 1 PM, 6 Mar 2003

1 An abstract manifold structure for $\mathbb{RP}^n$

On HW1, problem 2, you looked at certain maps ϕ_i (where $i = x, y, z, \ldots$) from $\mathbb{RP}^n$ to $\mathbb{R}^n$. The image of ϕ_z was called V_z in that problem; here it will serve as one of the "pages" in our atlas for an abstract manifold X (i.e., instead of the pages being called $U_1, U_2, \ldots$, they'll be called $V_x, V_y, \ldots$).

Given this set of pages, the functions ψ_{ij} defined in that problem can be used as transition functions in the definition of an abstract manifold.

A few details are missing.

(a) Describe all the sets $V_{i,j}$, including those for which $i = j$, and all the transition functions $\psi_{i,j}$. Explain why the transition functions satisfy the cocycle condition.

Let X denote the quotient space $\bigsqcup V_i / \sim$, where $\sqcup$ denotes the disjoint union of sets, and "$\sim$" is the equivalence relation defined by the $\psi_{i,j}$s.

(b) Show that the collection of maps ϕ_i described in the problem can be used to define a function Φ from $\mathbb{RP}^n$ to X that makes the following diagram commutative for each i:

$$\begin{array}{ccc} U_i & \xrightarrow{\subset} & \mathbb{RP}^n \\ {\scriptstyle \phi_i}\downarrow & & \downarrow{\scriptstyle \Phi} \\ V_i & \xrightarrow{x \mapsto [x]} & X \end{array}$$

Remark: The map Φ above is a homeomorphism between $\mathbb{RP}^n$ and the "abstract manifold" we built. Showing this involves giving $\mathbb{RP}^n$ a topology (the quotient topology) and the proving that each ϕ_i is a local homeomorphism. (Which you need not do for this problem!)

Because each of the transition functions ψ_{ij} is a smooth function, you can define tangent vectors (as you did in the last homework). I'll recommend a general definition of a tangent vector for an abstract manifold. I'll write it my the usual notation for abstract manifolds, where the sets are called

U_i; to apply it in this particular case, replace U_i with V_i throughout the definition:

If $(\{U_i\}, \{\psi_{ij}\})$ is a collection that defines an abstract k-manifold X, then a *point* of X is an equivalence class of points in the disjoint union of the sets $U_i \subset R^k$. We'll define the notion of a *tangent vector at p*.

For any point $p_i \in U_i$, the tangent vectors to R^k at the point p_i can be described as ordered pairs $(p_i; \mathbf{v})$ where $\mathbf{v}$ is an element of $\mathbb{R}^k$ (considered as a vector space). We'll now define an equivalence relation on the set of all such tangent vectors at all points of all the sets U_i, namely, for $p_i \in U_i$ and $p_j \in U_j$, we say

$$(p_i; \mathbf{v}) \approx (p_j, \mathbf{w})$$

if and only if

$$p_i \sim p_j \text{ and } d\psi_{ij}(p_i)(\mathbf{v}) = \mathbf{w}.$$

In other words, for $(p_i; \mathbf{v})$ to be the same as $(p_j; \mathbf{w})$, we need that p_i and p_j represent the same point in X, i.e., that $\psi_{ij}(p_i) = p_j$, *and* that the derivative map for ψ_{ij} at p_i takes $\mathbf{v}$ to $\mathbf{w}$.

(c) In $\mathbb{RP}^2$, consider the point $[1, 1, 2]$. Under the map ϕ_x, this corresponds to the point $(1, 2) \in V_x$; under ϕ_z it corresponds to the point $(1/2, 1/2) \in V_z$. Consider the vector

$$((1, 2); \begin{bmatrix} 2 \\ 7 \end{bmatrix})$$

which is a tangent vector at the point $(1, 2) \in V_x$. What's the corresponding tangent vector in V_z? In V_y?

(d) Now consider the vector

$$((3, 2); \begin{bmatrix} 2 \\ 7 \end{bmatrix})$$

which is a tangent vector at the point $(3, 2) \in V_x$. What's the corresponding tangent vector in V_z? In V_y?

Remark: this problem shows that in general the correspondence between tangent vectors in the V_js varies from point to point. On the other hand, if all the transition functions ψ_{ij} were translations, as discussed above, or even linear transformations, then the correspondence would *not* vary with the basepoint.