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OPTIMAL UPWARD PLANARITY TESTING OF SINGLE-SOURCE DIGRAPHS*

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Abstract. A digraph is upward planar if it has a planar drawing such that all the edges are monotone with respect to the vertical direction. Testing upward planarity and constructing upward planar drawings is important for displaying hierarchical network structures, which frequently arise in software engineering, project management, and visual languages. In this paper we investigate upward planarity testing of single-source digraphs; we provide a new combinatorial characterization of upward planarity and give an optimal algorithm for upward planarity testing. Our algorithm tests whether a single-source digraph with n vertices is upward planar in $O(n)$ sequential time, and in $O(\log n)$ time on a CRCW PRAM with $n \log\log n / \log n$ processors, using $O(n)$ space. The algorithm also constructs an upward planar drawing if the test is successful. The previously known best result is an $O(n^2)$-time algorithm by Hutton and Lubiw [*Proc. 2nd ACM–SIAM Symposium on Discrete Algorithms*, SIAM, Philadelphia, 1991, pp. 203–211]. No efficient parallel algorithms for upward planarity testing were previously known.

Key words. graph drawing, planar graph, upward drawing, triconnected components, parallel algorithm

AMS subject classifications. 68Q20, 68R10, 68U05, 05C10, 06A99

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1. Introduction.

The upward planarity of digraphs is a fundamental issue in the area of graph drawing and has been extensively investigated. A digraph is upward planar if it has a planar upward drawing, i.e., a planar drawing such that all the edges are monotone with respect to the vertical direction (see Figure 1a). Planarity and acyclicity are necessary but not sufficient conditions for upward planarity, as shown in Figure 1b.

Testing upward planarity and constructing upward planar drawings are important for displaying hierarchical network structures, which frequently arise in a wide variety of areas. Key areas of application include software engineering, project management, and visual languages. Especially significant in a number of applications are single-source digraphs, such as subroutine-call graphs, is-a hierarchies, and organization charts. Also, upward planarity of single-source digraphs has deep combinatorial

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implications in the theory of ordered sets. Namely, the orders defined by the transitive closure of upward planar single-source digraphs have bounded dimension [34] so that they can be compactly represented.

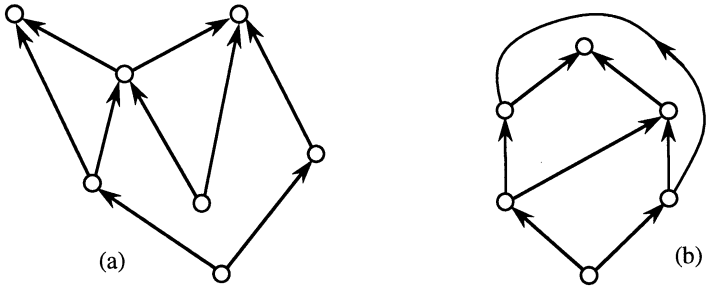

FIG. 1. *Examples of planar acyclic digraphs:* (a) *upward planar;* (b) *not upward planar.*

A survey on algorithms for planarity testing and graph drawing can be found in [7]. Previous work on upward planarity is as follows.

Combinatorial results on upward planarity for covering digraphs of lattices were first given in [22, 26]. Further results on the interplay between upward planarity and ordered sets are surveyed by Rival [30]. Lempel, Even, and Cederbaum [23] relate the planarity of biconnected undirected graphs to the upward planarity of st-digraphs. A combinatorial characterization of upward planar digraphs is provided in [21, 9]; namely, a digraph is upward planar if and only if it is a subgraph of a planar st-digraph.

Di Battista, Tamassia, and Tollis [9, 12] give algorithms for constructing upward planar drawings of st-digraphs and investigate area bounds and symmetry display. Tamassia and Vitter [32] show that the above drawing algorithms can be efficiently parallelized. Upward planar drawings of trees and series-parallel digraphs are studied in [29, 31, 6, 13, 15] and [1, 2], respectively.

In [8] it is shown that for the special case of bipartite digraphs, upward planarity is equivalent to planarity. In [3, 4] a polynomial-time algorithm is given for testing the upward planarity of digraphs with a prescribed embedding. Thomassen [33] characterizes the upward planarity of single-source digraphs in terms of forbidden circuits. Hutton and Lubiw [19] combine Thomassen's characterization with a decomposition scheme to test the upward planarity of an n-vertex single-source digraph in $O(n^2)$ time. Very recently, Papakostas [25] has given a polynomial-time algorithm for upward planarity testing of outerplanar digraphs, and Garg and Tamassia [16] have shown that upward planarity testing is NP-complete for general digraphs.

In this paper we investigate upward planarity testing of single-source digraphs. Our main results are summarized as follows:

- We provide a new combinatorial characterization of upward planarity within a given embedding in terms of a forest embedded in the face-vertex incidence graph.
- We reduce the upward planarity testing problem to that of finding a suitable orientation of a tree that synthetically represents the decomposition of a graph into its triconnected components.
- We show that the above combinatorial results yield an optimal $O(n)$-time upward planarity testing algorithm for single-source digraphs. The algorithm also constructs an upward planar drawing if the test is successful. Our algorithm is an improvement over the previously known best result [19] by an $O(n)$ factor in the time

complexity. Our algorithm is easy to implement and does not require any complex data structure.

- We efficiently parallelize the above algorithm to achieve $O(\log n)$ time on a CRCW PRAM with $n \log\log n / \log n$ processors. Hence, we provide the first efficient parallel algorithm for upward planarity testing. Our parallel time complexity is the same as that of the best parallel algorithm for planarity testing [28, 27].
- Finally, as a side effect we provide an optimal parallel algorithm for testing acyclicity of a planar n-vertex single-source digraph in $O(\log n)$ time with $n/\log n$ processors on an EREW PRAM.

Open problems include the following:

- devising efficient dynamic algorithms for upward planarity testing of single-source digraphs;
- exploring the area requirements of upward planar drawings of single-source digraphs;
- reducing the time complexity of upward planarity testing of planar digraphs with a prescribed embedding; and
- identifying additional classes of planar digraphs for which upward planarity can be tested in polynomial time.

The remainder of this paper is organized as follows. Section 2 contains preliminary definitions and results. The problem of testing upward planarity for planar single-source digraphs with a prescribed embedding is investigated in section 3. A combinatorial characterization of upward planarity for single-source digraphs is given in sections 4, 5, and 6. The complete upward planarity testing algorithm for single-source digraphs is presented in section 7. Also in section 7, two examples of application of the algorithm are illustrated. In the first example the considered digraph is not upward drawable; in the second example an upward drawable digraph is considered.

2. Preliminaries. In this section we recall some terminology and basic results on upward planarity. We also review the SPQR-tree, introduced in [10, 11], and the combinatorial characterization of upward planarity for embedded planar digraphs, shown in [3, 4]. We assume the reader's familiarity with planar graphs.

2.1. Drawings and embeddings. A drawing of a graph maps each vertex to a distinct point of the plane and each edge (u, v) to a simple Jordan curve with endpoints u and v. A *polyline* drawing maps each edge into a polygonal chain. A *straight-line* drawing maps each edge into a straight-line segment.

A drawing is planar if no two edges intersect except, possibly, at common endpoints. A graph is planar if it has a planar drawing. Two planar drawings of a planar graph G are equivalent if, for each vertex v, they have the same circular clockwise sequence of edges incident on v. Hence, the planar drawings of G are partitioned into equivalence classes. Each such class is called an *embedding* of G. An embedded planar graph is a planar graph with a prescribed embedding. A triconnected planar graph has a unique embedding up to a reflection. A planar drawing divides the plane into topologically connected regions delimited by circuits, called faces. The external face is the boundary of the unbounded region. Two drawings with the same embedding have the same faces. Hence, one can speak of the faces of an embedding.

Let G be a digraph, i.e., a directed graph. A *source* of G is a vertex without incoming edges. A *sink* of G is a vertex without outgoing edges. An *internal vertex* of G has both incoming and outgoing edges. An *sT-digraph* is an acyclic digraph with exactly one source.

Let f be a face of planar drawing (or embedding) of a digraph. A *source-switch* (*sink-switch*) of f is a source (sink) of f. Note that a *source-switch* (*sink-switch*) is not necessarily a source (sink) of G.

An *upward* drawing of a digraph is such that all the edges are represented by directed curves increasing monotonically in the vertical direction. A digraph has an upward drawing if and only if it is acyclic. A digraph is *upward planar* if it admits a planar upward drawing. Note that a planar acyclic digraph does not necessarily have a planar upward drawing, as shown in Fig. 1b. An upward planar digraph also admits a planar upward straight-line drawing [21, 9]. A planar st-digraph is a planar digraph with exactly one source s and one sink t, connected by edge (s,t). A digraph is upward planar if and only if it is a subgraph of a planar st-digraph [21, 9].

A planar embedding of a digraph is *candidate* if the incoming (outgoing) edges around each vertex are consecutive. The planar embedding underlying an upward drawing is candidate.

An *upward embedding* of a digraph G is an embedding of G such that

- each source- and sink-switch of each face of G is labeled *small* or *large*;
- there exists a planar straight-line upward drawing of G where each switch labeled *small* corresponds to an angle with measure $< \pi$, and each switch labeled *large* has measure $> \pi$.

Finally, the following lemma is due to Hutton and Lubiw.

LEMMA 1 (see [19]). *If a digraph has a single source, then it is upward planar if and only if its biconnected components are upward planar.*

Due to this result, in the remainder of the paper we will consider only biconnected digraphs.

2.2. SPQR-trees. In the following we summarize SPQR-trees. For more details see [10, 11]. SPQR-trees are closely related to the classical decomposition of biconnected graphs into triconnected components [17].

Let G be a biconnected graph. A *split pair* of G is either a separation pair or a pair of adjacent vertices. A *split component* of a split pair $\{u,v\}$ is either an edge (u,v) or a maximal subgraph C of G such that C contains u and v, and $\{u,v\}$ is not a split pair of C. Note that a vertex w distinct from u and v belongs to exactly one split component of $\{u,v\}$.

Let $\{s,t\}$ be a split pair of G. A *maximal split pair* $\{u,v\}$ of G with respect to $\{s,t\}$ is a split pair of G distinct from $\{s,t\}$ such that, for any other split pair $\{u',v'\}$ of G, there exists a split component of $\{u',v'\}$ containing vertices u, v, s, and t.

Let $e(s,t)$ be an edge of G, called *reference edge*. The *SPQR-tree* $\mathcal{T}$ of G with respect to e describes a recursive decomposition of G induced by its split pairs. Tree $\mathcal{T}$ is a rooted ordered tree whose nodes are of four types: S, P, Q, and R. Each node μ of $\mathcal{T}$ has an associated biconnected multigraph, called the *skeleton* of μ, and denoted by *skeleton*(μ). Also, it is associated with an edge of the skeleton of the parent ν of μ, called the *virtual edge* of μ in *skeleton*(ν). Tree $\mathcal{T}$ is recursively defined as follows.

Trivial case. If G consists of exactly two parallel edges between s and t, then $\mathcal{T}$ consists of a single Q-node whose skeleton is G itself.

Parallel case. If the split pair $\{s,t\}$ has at least three split components $G_1,\ldots,G_k$ ($k \geq 3$), the root of $\mathcal{T}$ is a P-node μ. Graph *skeleton*(μ) consists of k parallel edges between s and t, denoted $e_1,\ldots,e_k$, with $e_1 = e$.

Series case. Otherwise, the split pair $\{s,t\}$ has exactly two split components, one of them is the reference edge e, and we denote with G' the other split component. If G' has cutvertices $c_1,\ldots,c_{k-1}$ ($k \geq 2$) that partition G into its blocks

$G_1, \ldots, G_k$, in this order from s to t, the root of $\mathcal{T}$ is an S-node μ. Graph *skeleton*(μ) is the cycle $e_0, e_1, \ldots, e_k$, where $e_0 = e$, $c_0 = s$, $c_k = t$, and e_i connects c_{i-1} with c_i $(i = 1, \ldots, k)$.

Rigid case. If none of the above cases applies, let $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ be the maximal split pairs of G with respect to $\{s, t\}$ $(k \geq 1)$, and for $i = 1, \ldots, k$, let G_i be the union of all the split components of $\{s_i, t_i\}$ except for the one containing the reference edge e. The root of $\mathcal{T}$ is an R-node μ. Graph *skeleton*(μ) is obtained from G by replacing each subgraph G_i with the edge e_i between s_i and t_i.

Except for the trivial case, μ has children $\mu_1, \ldots, \mu_k$ in this order such that μ_i is the root of the SPQR-tree of graph $G_i \cup e_i$ with respect to reference edge e_i $(i = 1, \ldots, k)$. Edge e_i is said to be the *virtual edge* of node μ_i in *skeleton*(μ) and of node μ in *skeleton*(μ_i). Graph G_i is called the *pertinent graph* of node μ_i, and of edge e_i.

The tree $\mathcal{T}$ so obtained has a Q-node associated with each edge of G, except the reference edge e. We complete the SPQR-tree by adding another Q-node, representing the reference edge e, and making it the parent of μ so that it becomes the root. Observe that we are defining SPQR-trees of graphs; however, the same definition can be applied to digraphs. An example of SPQR-tree is shown in Figure 2.

Let μ be a node of $\mathcal{T}$. We have the following:

- if μ is an R-node, then *skeleton*(μ) is a triconnected graph;
- if μ is an S-node, then *skeleton*(μ) is a cycle;
- if μ is a P-node, then *skeleton*(μ) is a triconnected multigraph consisting of a bundle of multiple edges;
- if μ is a Q-node, then *skeleton*(μ) is a biconnected multigraph consisting of two multiple edges.

The skeletons of the nodes of $\mathcal{T}$ are homeomorphic to subgraphs of G. The SPQR-trees of G, with respect to different reference edges, are isomorphic and are obtained one from the other by selecting a different Q-node as the root. Hence, we can define the *unrooted SPQR*-tree of G without ambiguity.

The SPQR-tree $\mathcal{T}$ of a graph G with n vertices and m edges has m Q-nodes and $O(n)$ S-, P-, and R-nodes. Also, the total number of vertices of the skeletons stored at the nodes of $\mathcal{T}$ is $O(n)$.

A graph G is planar if and only if the skeletons of all the nodes of the SPQR-tree $\mathcal{T}$ of G are planar. An SPQR-tree $\mathcal{T}$ rooted at a given Q-node represents all the planar drawings of G having the reference edge (associated with the Q-node at the root) on the external face (see Figure 2). Namely, such drawings can be constructed by the following recursive procedure:

- construct a drawing of the skeleton of the root ρ with the reference edge of the parent of ρ on the external face;
- for each child μ of ρ
  - let e be the virtual edge of μ in *skeleton*(ρ), and let H be the pertinent graph of μ plus edge e;
  - recursively draw H with the reference edge e on the external face;
  - in *skeleton*(ρ), replace virtual edge e with the above drawing of H minus edge e.

2.3. Upward planarity testing of embedded digraphs. In the remainder of this section we recall the combinatorial characterization of upward planarity for planar digraphs with a fixed embedding, given in [3, 4], which will be used extensively in this paper.

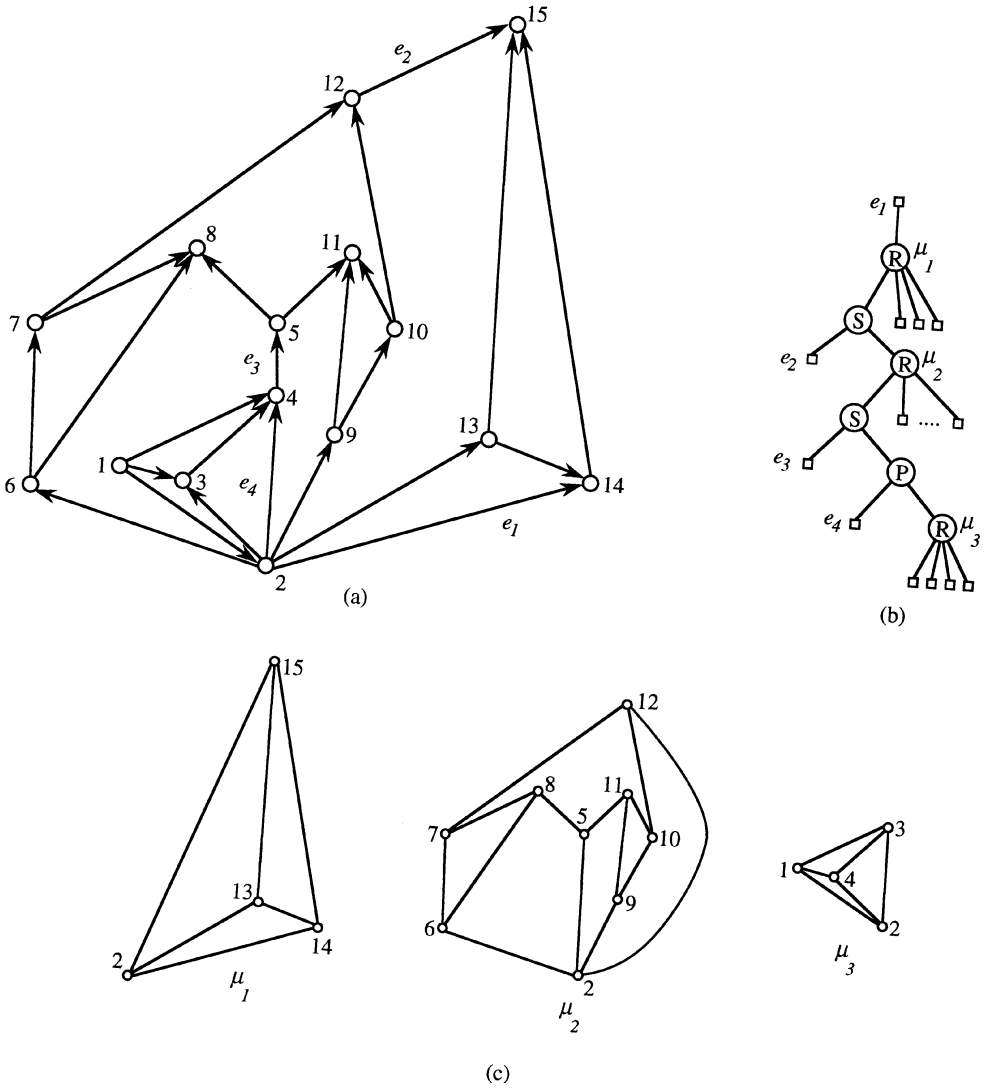

FIG. 2. (a) *A planar biconnected digraph G.* (b) *SPQR-tree $\mathcal{T}$ of G, where the* Q*-nodes are represented by squares.* (c) *Skeletons of the* R*-nodes.*

In [3, 4], the problem of testing whether an embedded planar digraph G admits a planar upward drawing is formulated as a perfect c-matching problem on a bipartite graph derived from G. To introduce this formulation we need some notation and definitions.

Let Γ be a planar straight-line upward drawing of an embedded upward planar digraph G. (As shown in [21, 9], every upward planar digraph admits a planar upward straight-line drawing.) We say that a sink t (source s) of G is *assigned* to a face f of Γ if the angle defined by the two edges of f incident on t (s) is greater than π. Informally speaking, t (s) is assigned to f if it "penetrates" into face f. Clearly, each sink (source) can be assigned only to one face, while an internal vertex is not assigned to any face. In [3, 4], it is shown that the number of vertices assigned to a face f in

any upward drawing is equal to the capacity $c(f)$ of the face itself, which is defined as follows. Let n_f be the number of sink-switches of f (n_f is also equal to the number of source-switches of f). We set $c(f) = n_f - 1$ if f is an internal face and $c(f) = n_f + 1$ if f is the external face. In the following, we associate to each vertex x and each face f the quantity $Z(x, f)$, where $Z(x, f) = 1$ if x is a switch of f, $Z(x, f) = 0$ otherwise. Clearly, we have that $2n_f = \sum_{x \in f} Z(x, f)$.

This intuitive idea of assignment of vertices to faces can be formally expressed as a perfect c-matching problem [24]. Namely, given a planar digraph G with a candidate planar embedding Ψ, we associate with G and Ψ the bipartite network $N(L_1, L_2, E_N)$ with vertex set $L_1 \cup L_2$ and edge-set E_N, where (i) the vertices of L_1 represent the sources and sinks of G; (ii) the vertices of L_2 represent the faces of Ψ; and (iii) E_N has an edge (v, f) if and only the vertex of G represented by $v \in L_1$ lies on the face of Ψ represented by $f \in L_2$. The c-matching problem for G and Ψ is described by the following equations:

$$\sum_{(v,f) \in E_N} x_{vf} = c(f), \quad \forall f \in L_2,$$

$$\sum_{(v,f) \in E_N} x_{vf} = 1, \quad \forall v \in L_1,$$

where $x_{vf} = 1$ indicates that vertex v is *assigned* to face f and $x_{vf} = 0$ indicates otherwise. A solution of this c-matching problem is called an *upward consistent assignment* of the variables x_{vf} and is denoted by $\mathcal{A}$. The equations of the first set are called *capacity equations*.

LEMMA 2 (see [3, 4]). *Let G be a digraph with a candidate planar embedding Ψ. Then Ψ is an upward embedding of G if and only if the c-matching problem associated with G and Ψ admits an upward consistent assignment.*

If $\mathcal{A}$ is an upward consistent assignment for Ψ, and f is a face of Ψ, we denote by $A(f)$ the set of vertices of G assigned to f in $\mathcal{A}$.

3. Embedded digraphs. In this section we give a new combinatorial characterization of upward planarity for planar single-source digraphs with a prescribed embedding. This characterization yields an optimal algorithm for testing whether an embedded planar single-source digraph has an upward planar drawing that preserves the embedding.

Given a planar single-source digraph G and an upward embedding Γ of G, from the first condition on the capacity equations of the perfect matching problem and from the fact that G has a unique source, the following properties can be easily derived (see Figure 3).

FACT 1. *The source of G is the bottommost vertex of Γ.*

FACT 2. *For the external face h of Γ, all the sink-switches are sinks of G and are assigned to h. (See Figure* 3a.*)*

FACT 3. *For each internal face f, at most one sink-switch (the topmost vertex of f in Γ) is not a sink of G and all but one sink switches are assigned to f. (See Figure* 3b.*)*

We shall also use the following result about cycles in planar single-source digraphs.

LEMMA 3. *Let G and G' be planar single-source digraphs such that G' is obtained from G by means of one of the following operations:*

- *adding a new vertex v and a new edge (u, v) or (v, u), connecting v to a vertex u of G;*

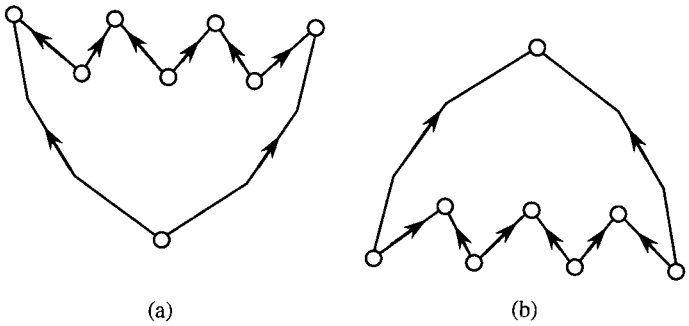

FIG. 3. *Schematic illustration of* (a) *the external face;* (b) *an internal face of an upward planar drawing of a planar single-source digraph.*

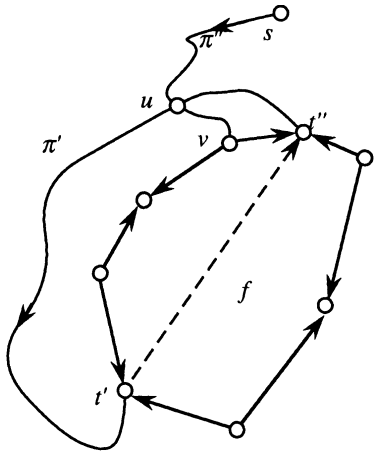

FIG. 4. *Illustration of the proof of Lemma* 3.

- *adding a directed edge between the source and a sink on the same face in some embedding of G;*
- *adding a directed edge between two sink-switches on the same face in some embedding of G.*

Then G is acyclic if and only if G' is acyclic.

Proof. The acyclicity is trivially preserved by the first two operations. Regarding the third operation, consider an embedding of G with the source on the external face, and assume, for a contradiction, that G is acyclic and that adding the edge (t', t'') between sink-switches t' and t'' of face f causes the resulting graph G' to have a cycle γ (see Figure 4). Cycle γ must consist of edge (t', t'') and a directed path π' in G from t'' to t'. Let v be the neighbor of t'' in f inside γ, and let π'' be a directed path from the source of G to v. Since the source is external to cycle γ, path π'' must have at least a vertex in common with path π'. Let u be the last vertex of π'' that is also on π'. We have that G has a cycle consisting of edge (v, t''), the subpath of π' from t'' to u, and the subpath of π'' from u to v, which is a contradiction. □

Given an embedded planar single-source digraph G, the *face-sink graph* F of G is the incidence graph of the faces and the sink-switches of G (see Figures 5a and 6).

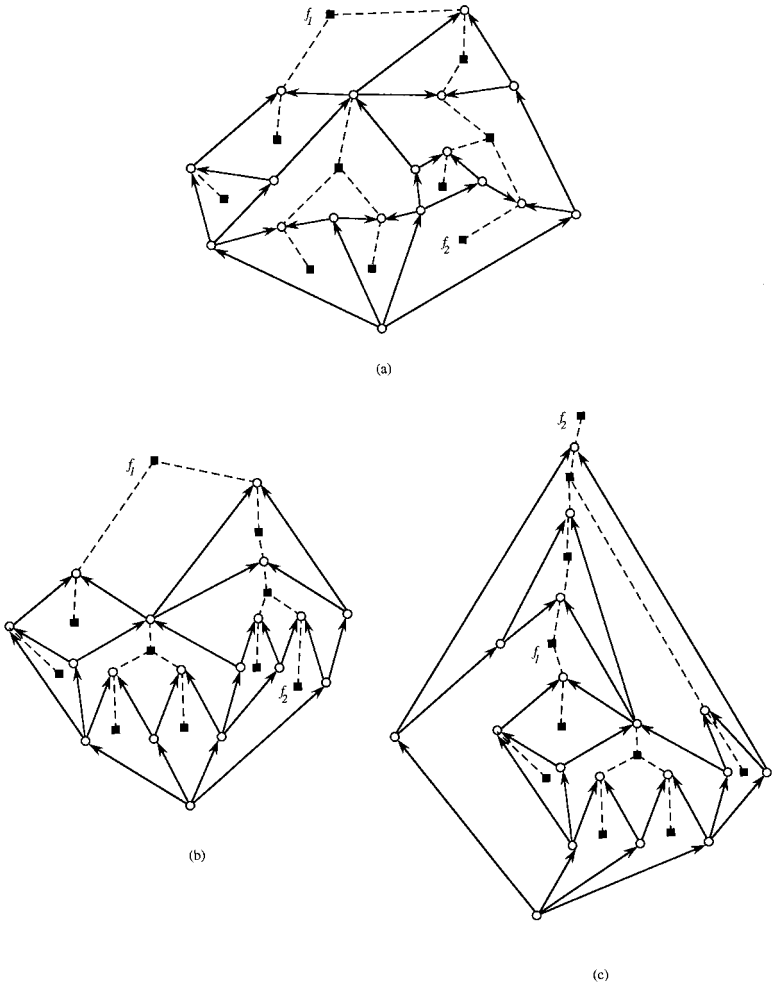

FIG. 5. (a) *An embedded planar single-source digraph G and its face-sink graph.* (b)–(c) *Upward drawings of G that preserve the embedding, with different external faces.*

Namely,

- the vertices of F are the faces and the sink-switches of G;
- graph F has an edge (f, v) if v is a sink-switch on face f.

THEOREM 1. *Let G be an embedded planar single-source digraph and let h be a face of G. Digraph G has an upward planar drawing that preserves the embedding with external face h if and only if all of the following conditions are satisfied:*

1. *graph F is a forest;*

2. *there is exactly one tree T of F with no internal vertices of G, while the remaining trees have exactly one internal vertex;*

3. *h is in tree T; and*

4. *the source of G is in the boundary of h.*

Also, if the above conditions are satisfied, then an embedded planar st-digraph G' containing G as an embedded subgraph is obtained as follows:

1. *root tree T at h and each remaining tree of F at its (unique) internal vertex;*

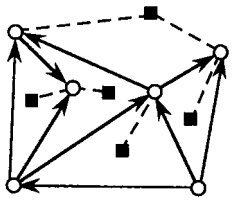

FIG. 6. *An embedded planar single-source digraph that does not have an upward drawing that preserves the embedding.*

2. *orient F by directing edges toward the roots;*
3. *prune the leaves from every tree of F; and*
4. *add the resulting forest $\hat{F}$ and the edge (s, h) to G.*

Proof. Only if. Let Γ be any planar upward drawing of G that preserves the original planar embedding and has external face h. By Fact 1, condition 4 is verified. Orient the face-sink graph F of G by directing edge (v, f) from v to f if v is a sink assigned to face f, and from f to v otherwise. By Facts 2–3, each vertex of F has at most one outgoing edge. Specifically, each internal face and each sink has exactly one outgoing edge, while each internal vertex and the external face have no outgoing edges. Now, label the vertices of F as follows: the label of a sink-switch is its y-coordinate in Γ; the label of an internal face f is $y(v) - \epsilon$, where v is the sink-switch not assigned to f, and ϵ is a suitably small positive real value, and the label of the external face h is $+\infty$. Since Γ is an upward drawing, the edges of F are directed by increasing labels. We conclude that F is a forest of sink-trees. One tree is rooted at h, while the other trees are rooted at internal vertices. This shows conditions 1–2. Condition 3 follows from Fact 1.

If. We show that, if F satisfies the conditions of the theorem, then G is a subgraph of a planar st-digraph G', which is obtained as the union of G and $\hat{F}$. This implies that G is upward planar. Planarity is preserved since a star is inserted in each face. Also, G' has exactly one source (s) and one sink (h) connected by a directed edge. It remains to be shown that G' is acyclic. By the construction of G' and Lemma 3, we have that G' is acyclic if and only if G is acyclic. Assume, for a contradiction, that G is not acyclic. Let γ be a cycle of G that does not enclose any other cycle. Note that the source s must be outside γ. If γ is a face of G, then F has an isolated vertex associated with face γ, which is a contradiction. Otherwise (γ is not a face of G), the subgraph $\hat{F}'$ of $\hat{F}$ enclosed by γ consists of a forest of trees, each with exactly one internal vertex. Let H be the digraph obtained from the subgraph of G enclosed by γ by removing the edges of γ, and adding a new vertex s' together with edges from s' to all the vertices of γ. By our choice of cycle γ, H is a planar single-source digraph. Adding $\hat{F}'$ to H yields a planar single-source digraph without sinks, and hence a digraph with cycles. By Lemma 3, H must also have cycles, which is again a contradiction. □

Theorem 1 is illustrated in Figures 5–6. The following algorithm tests whether an embedded planar single-source digraph G is upward planar, and reports all the faces of G that can be external in an upward planar drawing of G with the prescribed embedding.

ALGORITHM. *Embedded-Test.*

1. Construct the face-sink graph F of G.
2. Check conditions 1 and 2 of Theorem 1. If these conditions are not verified,

then return "not-upward-planar" and stop.

3. Report the set of faces of G that contain vertex s in their boundaries and are associated with nodes of tree T. If such a set of faces is empty, then return "not-upward-planar"; else return "upward-planar."

For the example of Figure 5, Algorithm *Embedded-Test* returns "upward-planar" and reports two faces.

THEOREM 2. *Let G be an embedded planar single-source digraph with n vertices. Algorithm* embedded test *determines whether G has an upward planar drawing that preserves the embedding and reports all the admissible external faces. It runs in $O(n)$ sequential time and in $O(\log n)$ time on a CRCW PRAM with $n \cdot \alpha(n)/\log n$ processors, using $O(n)$ space.*

Proof. The correctness of the algorithm follows directly from Theorem 1. All the steps can be performed sequentially in $O(n)$ time with straightforward methods.

Regarding the parallel complexity, steps 1 and 3 take $O(\log n)$ time on a CREW PRAM with $n/\log n$ processors, using list-ranking [5]. Step 2 can be executed by computing a spanning forest of the face-sink graph, which takes $O(\log n)$ time on a CRCW PRAM with $n \cdot \alpha(n)/\log n$ processors [5], and thus determines the parallel time complexity. □

4. Upward planarity and SPQR-trees. Let G be a biconnected single-source digraph. In this section we give a combinatorial characterization of the upward planarity of G using SPQR-trees.

4.1. Basic definitions and main result. A digraph is *expanded* if each internal vertex has exactly one incoming edge or one outgoing edge. The *expansion* of a digraph is obtained by replacing each internal vertex v with two new vertices v_1 and v_2, which inherit the incoming and outgoing edges of v, respectively, and the edge (v_1, v_2). Observe that a digraph is acyclic if and only if its expansion is acyclic. A planar embedding of an expanded digraph is candidate. In the remainder of this section we consider only expanded digraphs because of the following property.

FACT 4. *A digraph is upward planar if and only if its expansion is upward planar.*

Let $G' = (V', E')$ and $G'' = (V'', E'')$ be two digraphs. The *union* of G' and G'', denoted by $G' \cup G''$, is a digraph $G = (V, E)$ with $V = V' \cup V''$ and $E = E' \cup E''$, i.e., G is obtained from G' and G'' by identifying the vertices in V' and V'' with common labels.

Let $\{u, v\}$ be a separation pair of G that decomposes G into p split components $J_1, \ldots, J_p$. We call a *component separated by* $\{u, v\}$ or simply a *component* any digraph obtained as the union of q of the split components in $\{J_1, \ldots, J_p\}$, with $0 < q < p$.

We call *peak* a digraph consisting of three vertices a, b, and t, and two directed edges (a, t) and (b, t). See Figure 7b.

Let K be a component of G with respect to the separation pair $\{u, v\}$. In the following, we denote with $G - K$ the digraph obtained from G by deleting every vertex belonging to K, except for the vertices u and v, and by K° the digraph obtained from K by deleting vertices u and v. Also, for simplicity we write $G - K_1 - K_2 - \cdots - K_m$ instead of $(\cdots((G - K_1) - K_2) \cdots - K_m)$.

Finally, we associate with a component K of G either a directed edge or a peak (see Figures 7a–b) according to the following rules.

Rule 1. u and v are sources of K: a peak with $a \equiv u$ and $b \equiv v$.

Rule 2. u is a source of K and v is a sink of K.

(a) $s \notin K^\circ$: a directed edge (u, v).

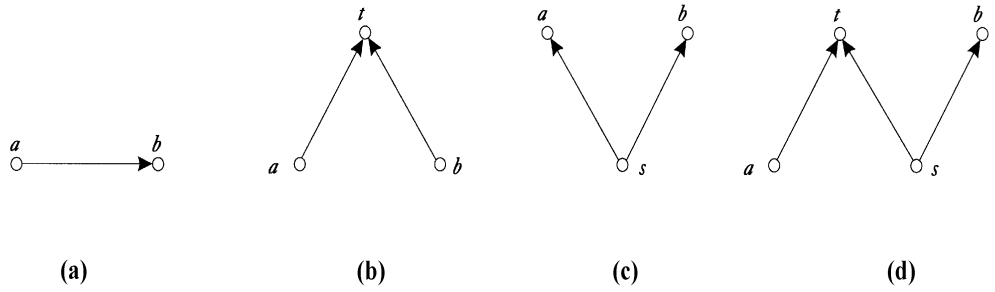

FIG. 7. (a) *Directed edge;* (b) *peak;* (c) *valley;* (d) *zig-zag.*

(b) $s \in K^\circ$: a peak with $a \equiv u$ and $b \equiv v$.

Rule 3. u is a source of K and v is an internal vertex of K.

(a) v is a source of $G - K$ and $s \notin K^\circ$: a directed edge (u, v).

(b) v is not a source of $G - K$ or $s \in K^\circ$: a peak with $a \equiv u$ and $b \equiv v$.

Rule 4. u and v are not sources of K.

(a) u is a source of $G - K$: a directed edge (u, v).

(b) u is not a source of $G - K$: a directed edge (v, u).

The digraph associated to K by the above rules is called *directed-virtual-edge* and will be denoted by $d(K, G - K)$. Observe that the choice of the directed-virtual-edge depends, in general, both on K and on $G - K$.

We call *minor* of G either G itself or the digraph

$$G - K_1 - \cdots - K_m \cup d(K_1, G - K_1) \cup \cdots \cup d(K_m, G - K_m),$$

where $K_1, \ldots, K_m, m \geq 1$, are components of G with the property that no two components share a common edge. In other words, the digraph $G - K_1 - \cdots - K_m \cup d(K_1, G - K_1) \cup \cdots \cup d(K_m, G - K_m)$ is obtained from G by replacing $K_1, \ldots, K_m$ with the corresponding directed-virtual-edges (see Figure 8). Observe that, in general, a minor of a minor of G is not a minor of G.

Let G be a planar single-source digraph, and let $\mathcal{T}$ be its SPQR-unrooted tree. The *sT-skeleton* of a node μ of $\mathcal{T}$, denoted by sT-skeleton(μ), is the minor of G obtained from the skeleton of μ by replacing each virtual edge with the directed-virtual-edge associated to its pertinent digraph. The *reference directed-virtual-edge* is the directed-virtual-edge associated with the pertinent digraph of the reference edge of μ. Examples of sT-skeletons are shown in Figure 9.

The main result of this section is summarized in the following theorem.

THEOREM 3. *A biconnected acyclic single-source digraph G is upward planar if and only if there exists a rooting of the SPQR-tree $\mathcal{T}$ of the expansion of G at a reference edge containing the source, such that the sT-skeleton of each node μ of $\mathcal{T}$ has a planar upward drawing with the reference directed-virtual-edge on the external face.*

The proof of Theorem 3 is given in the next two sections. Section 5 shows the *only-if* part, while section 6 shows the *if* part. Here we give some preliminary lemmas that will be used in the next sections.

4.2. Preliminary lemmas. In the following, S_G, T_G, and I_G will denote the set of sources, sinks, and internal vertices of G, respectively. If G has exactly one source, we denote such source by $s(G)$ (or simply by s, when no confusion arises).

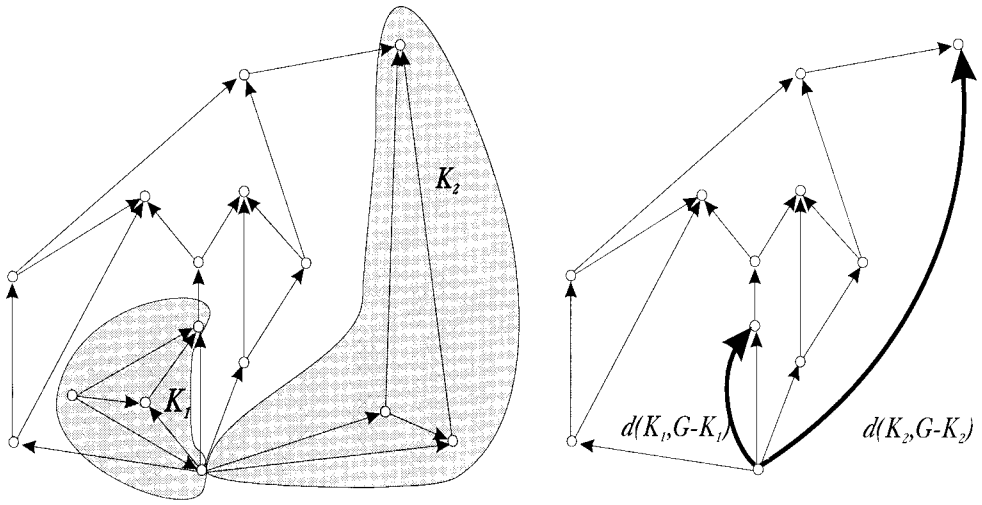

FIG. 8. *Construction of a minor.*

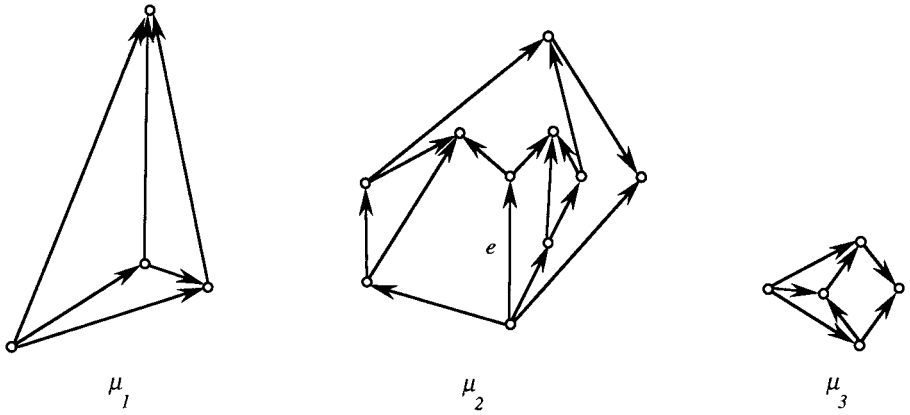

FIG. 9. *The sT-skeletons of the* R*-nodes of the digraph of Figure* 2.

We will make use of the following operations defined on an edge $e = (x, y)$ of a digraph G:

- *Contraction* (denoted by G/e) transforms G into a digraph G' obtained from G by removing edge e and by identifying vertices x and y.
- *Direct subdivision* transforms G into a digraph G' obtained from G by removing edge e and by adding a vertex z and edges (x, z) and (z, y).

We say that digraphs G_1 and G_2 are *homeomorphic* if both can be obtained by performing a finite number of direct subdivisions of a digraph G. Observe that we can have $G_1 = G$ or $G_2 = G$.

We call *valley* a digraph consisting of three vertices s', a, and b, and two directed edges (s', a) and (s', b) (see Figure 7c). Also, we call *zig-zag* a digraph consisting of four vertices s', t, a, and b, and three directed edges (s', t), (s', b), and (a, t) (see Figure 7d).

We denote by $x \to y$ a path from vertex x to vertex y. A vertex u is said to be *dominated* by vertex v if there is a path $v \to u$. We say that vertices x and y are

incomparable in G, denoted by $x \parallel y$, if there exists in G neither a path $x \to y$ nor a path $x \to y$.

FACT 5. *Let G be an sT-digraph. Then every vertex of G is dominated by s.*

FACT 6. *Let G be an acyclic digraph. G contains a source.*

LEMMA 4 (see [19]). *If G is an sT-digraph with $u \parallel v$ in G, then there exists in G a subgraph homeomorphic to a valley, with $a \equiv u$ and $b \equiv v$.*

LEMMA 5 (see [19]). *Let G be a connected acyclic digraph with exactly two sources u and v. Then there exists in G a subgraph homeomorphic to a peak, with $a \equiv u$ and $b \equiv v$.*

FACT 7. *Let G be a biconnected digraph and let $\{u, v\}$ be a separation pair of G. Neither u nor v is a cut-vertex of any component of G with respect to $\{u, v\}$.*

LEMMA 6. *Let G be an sT-digraph and let $\{u, v\}$ be a separation pair of G. Let K be a component with respect to $\{u, v\}$ such that $v \in I_K$ and K has exactly one source u. Then digraph K contains a subgraph homeomorphic to a peak, with $a \equiv u$ and $b \equiv v$.*

Proof. Let w be a vertex of K such that there exist two vertex disjoint directed paths $u \to w$ and $v \to w$ contained in K. The subgraph of K consisting of all edges and vertices belonging to the two paths is homeomorphic to a peak. Suppose w does not exist. Let V_v be the set of all vertices dominated by v in K (except v).

Let $\bar{V}_v$ be the set of all vertices not dominated by v in K. Observe that both V_v and $\bar{V}_v$ are nonempty and disjoint. Also, $V_v \cup \bar{V}_v \cup \{v\}$ is the set of vertices of K. Since v is not a cut-vertex of K, there is an edge connecting a vertex x of V_v with a vertex y of $\bar{V}_v$. If such an edge is (x, y), then y belongs to V_v, which is a contradiction. If such an edge is (y, x), then x is dominated by both v and u, and $x = w$, which is a contradiction. □

Finally, the following fact and lemmas concerning the embeddings of G will be used in the proof of the main theorem.

FACT 8. *Let G be a digraph and let G' be a digraph homeomorphic to a subgraph G'' of G. We have that*

(i) *if G is acyclic, then G' is acyclic.*
(ii) *if G is expanded, then G' is expanded.*
(iii) *if G is upward planar, then G' is upward planar.*
(iv) *if G'' is an sT-digraph, then G' is an sT-digraph.*

LEMMA 7 (see [19]). *Let G be a digraph, let (u, v) be an edge of G, and let vertex $u(v)$ have out-degree (in-degree) 1 in G. Let $G' = G/(u, v)$. We have that*

(i) *if G is acyclic then G' is acyclic.*
(ii) *if G is upward planar, then G' is upward planar.*

LEMMA 8. *Let G be an upward planar sT-digraph, and let Ψ_G be an upward embedding of G. Let $e = (s, u)$ be an edge of G embedded on the external face α of Ψ_G. Let $G' = G - e \cup P$ where P is a valley, i.e., P is the path $\{(s', s), (s', u)\}$. Then G' is an sT-digraph and has an upward embedding $\Psi_{G'}$, with P embedded on the external face.*

Proof. From Ψ_G we simply derive a candidate planar embedding $\Psi_{G'}$ of G' by replacing the edge (s, u) of G with the path P (see Figure 10).

We now show that $\Psi_{G'}$ is an upward embedding. This is done by deriving an upward consistent assignment $\mathcal{A}'$ associated to $\Psi_{G'}$ from the upward consistent assignment $\mathcal{A}$ associated to Ψ_G.

Let β be the internal face of Ψ_G containing the edge (s, u). We denote by $\gamma_1, \ldots, \gamma_p$ the faces of Ψ_G different from α and β.

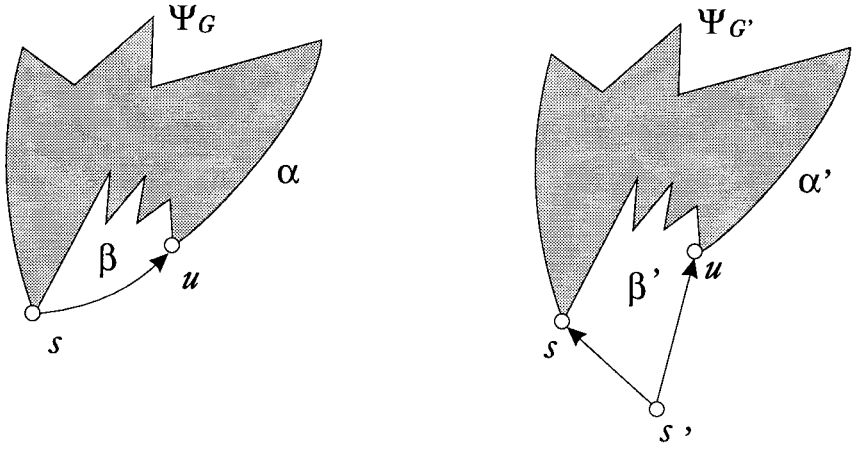

FIG. 10. *Construction of* $\Psi_{G'}$.

Clearly, there is a one-to-one correspondence between the faces of Ψ_G and the faces of $\Psi_{G'}$. We denote by α', β', $\gamma'_1, \ldots, \gamma'_p$ the faces of $\Psi_{G'}$ corresponding to $\alpha, \beta, \gamma_1, \ldots, \gamma_p$.

Also, it is $\gamma'_i = \gamma_i$, for $i = 1, \ldots, p$, and thus $c(\gamma'_i) = c(\gamma_i)$, for $i = 1, \ldots, p$. We show that $c(\alpha') = c(\alpha)$ and $c(\beta') = c(\beta)$.

Following the notation introduced in section 2.3, we have that $2n_{\alpha'} = 2n_\alpha - Z(s, \alpha) + Z(s, \alpha') - Z(u, \alpha) + Z(u, \alpha') + Z(s', \alpha')$. Since the edge (s, u) of α has been replaced by the edge (s', u) of α', it is $Z(u, \alpha') = Z(u, \alpha)$. It is easy to see that $Z(s, \alpha) = 1$, $Z(s, \alpha') = 0$, and $Z(s', \alpha') = 1$. Thus, $2n_{\alpha'} = 2n_\alpha$ and then $c(\alpha') = c(\alpha)$. In the same fashion, we can prove that $c(\beta') = c(\beta)$.

Let S (S') and T (T') be the set of sources and sinks of G (G'), respectively. Clearly $S' = S - \{s\} \cup \{s'\}$, and $T' = T$. Observe that, by Theorem 1, $s \in A(\alpha)$. $\mathcal{A}'$ is derived from $\mathcal{A}$ in the following way:

- $A'(\gamma'_i) = A(\gamma_i)$ for $i = 1, \ldots, p$;
- $A'(\alpha') = A(\alpha) - \{s\} \cup \{s'\}$; and
- $A'(\beta') = A(\beta)$.

It is straightforward to prove that $|A'(f)| = c(f)$ for each face $f \in \Psi_{G'}$. Since $\Psi_{G'}$ is a candidate embedding and $\mathcal{A}'$ is upward consistent, by Lemma 2, G' is upward planar. □

5. Proof of necessity for Theorem 3.

LEMMA 9. *Let G be a planar expanded sT-digraph G, $\{u, v\}$ a separation pair of G, and K a component with respect to $\{u, v\}$. Let $H = G - K$ and let $d_K = d(K, H)$ be the directed-virtual-edge associated to K with respect to G. Finally, let H' be the minor $H \cup d_k$. We have that*

(i) *H' is an expanded, acyclic sT-digraph.*

(ii) *if G is upward planar, then H' is upward planar.*

(iii) *if G is upward planar and $s(G) \in K^\circ$, then H' has an upward embedding with d_k on the external face.*

Proof. The following cases are possible, each corresponding to one of Rules 1–4 (in each case, the proof of (iii) is trivial and thus omitted).

1. u and v are sources of K.

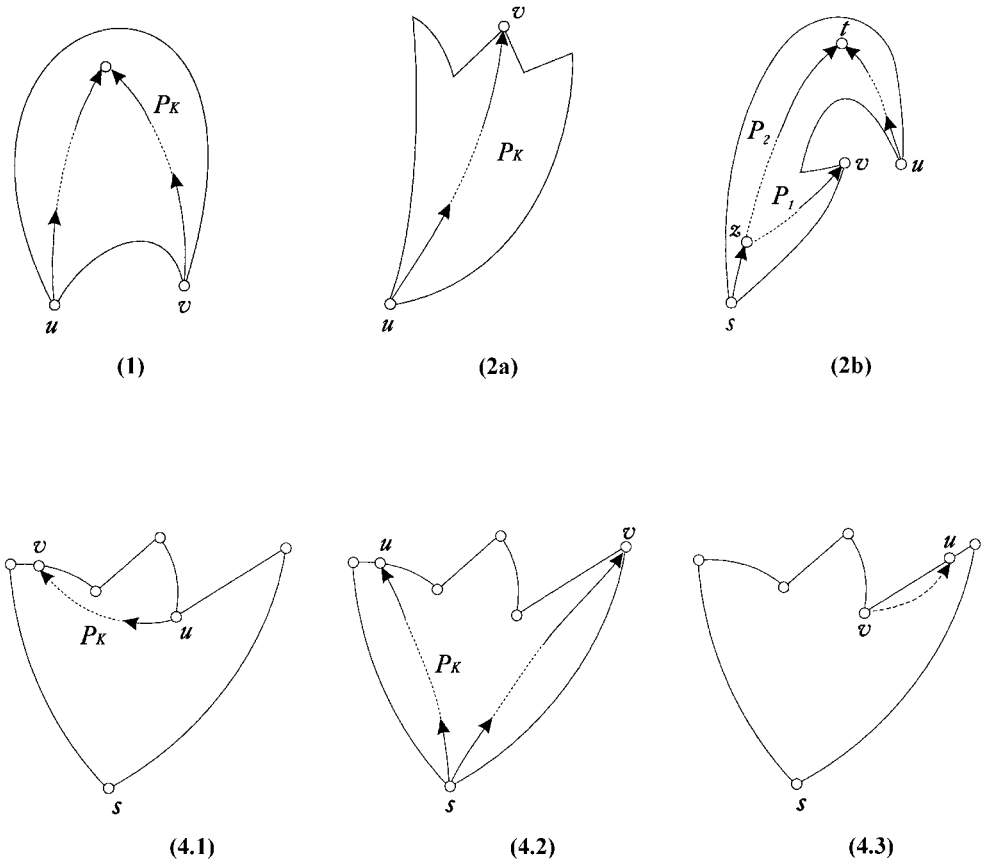

FIG. 11. *Various cases in the proof of Lemma* 9.

By Lemma 5, there is in K a path P_K homeomorphic to a peak, with $a \equiv u$ and $b \equiv v$ (see Figure 11). Thus, $H' = H \cup d_K$ is homeomorphic to a subgraph of G.

(i) By Fact 8, H' is an expanded acyclic digraph. We show now that it contains only one source. Vertices u and v are not both sources in H; otherwise G would contain two sources. Suppose one of u, v (say, u) is a source of H. Then $u \equiv s(G)$ and $u \equiv s(H')$; thus u is the only source of H'. Suppose neither u nor v is a source in H; then $s(G) \in H^\circ$ and s is the only source of H'.

(ii) By Fact 8, H' is upward planar.

2(a) u is a source of K and v is a sink of K and $s \notin K^\circ$.

Since $s \notin K^\circ$, u is the only source of K and there exists in K a path $u \to v$. Thus there is in K a path P_K homeomorphic to the directed edge (u, v), and then there is in G a subgraph homeomorphic to $H' = H \cup d_K$ (see Figure 11).

(i) By Fact 8, H' is an acyclic expanded digraph. Because of the existence of edge (u, v), v is not a source of H'. If $u \in S_H$, then $s(G) \equiv u$; thus u is the only source of H'. If $u \notin S_H$, $s(G) \in H^\circ$, and $s(G) \equiv s(H')$.

(ii) By Fact 8, H' is upward planar.

2(b) u is a source of K and v is a sink of K and $s \in K^\circ$.

Since $s \in K^\circ$, $s \neq u$ and, by Lemma 5, there is a vertex t distinct from s and u,

and two vertex disjoint paths $s \to t$ and $u \to t$. Note that u is not a source of H; otherwise u is a source of G, which is a contradiction. So v is the only source of H and since there is in G a path $s \to v$, we have that there is a path $s \to v$ in K. Moreover, since v is the only source of H, there is a path $v \to u$ in H and thus there is a path $v \to u$ in G. This implies that the path $s \to v$ and the path $u \to t$ are node-disjoint; otherwise there would be a path $u \to v$ and G would contain a cycle, which is a contradiction.

Let P_1 and P_2 be a path $s \to v$ and a path $s \to t$, respectively. Let z be the last vertex common to paths P_1 and P_2. Note that $z \neq t$, since P_1 and any path $u \to t$ are disjoint. Thus there exists in K a path P_K homeomorphic to a zig-zag, and then there is in G a subgraph homeomorphic to $H'' \equiv H \cup (z, v) \cup (z, t) \cup (u, t)$ (see Figure 11).

Observe that in H'', vertex v has in-degree 1. If we contract the edge (z, v) we obtain $H''/(z, v) \equiv H \cup d_K = H'$.

(i) By Fact 8 H'' is acyclic and expanded; thus, by Lemma 7 H' is acyclic. Note that every vertex except v has the same in- and out-degrees in H'' and H'. Since H'' is expanded and v is a source of H', H' is an expanded digraph. Moreover, since v is the only source of H, v is the only source of H'.

(ii) By Fact 8 H'' is upward planar; thus, by Lemma 7 H' is upward planar.

3(a) u is a source of K and v is an internal vertex of K; v is a source of $G - K$ and $s \notin K^\circ$.

Since u is the only source of K, there is a path $u \to v$ in K and the proof is analogous to that of case 2(a).

3(b) u is a source of K and v is an internal vertex of K; v is not a source of $G - K$ or $s \in K^\circ$.

1. If $s \in K^\circ$, then $s \neq u$ and K contains two sources; the proof is as in case 2(b).

2. If $s \notin K^\circ$, then u is the only source of K and, by Lemma 6, K contains a path P_K homeomorphic to a peak, with $u \equiv a$ and $v \equiv b$. Then $H' = H \cup d_K$ is homeomorphic to a subgraph of G.

(i) By Fact 8 H' is an expanded, acyclic digraph. If $u = s(G)$, then vertex u is the only source of H'. If $u \neq s(G)$, then u is not a source of H (otherwise G contains two sources, which is a contradiction), and $s(G)$ is the only source of H'. Thus, H' is an sT-digraph.

(ii) By Fact 8 H' is upward planar.

4. u and v are not sources of K.

Since u and v are not sources of K, we have that $s(G) \in K^\circ$. Then either u or v (or both) is a source of H. We discuss only the case where u is a source of H and Rule 4(a) is applied. In fact, if u is not a source of H, then v is a source of H and Rule 4(b) is applied; that is, the roles of u and v are interchanged. Three cases are possible.

1. If there exists a path $u \to v$ in K, then there is in K a path P_K homeomorphic to the directed edge (u, v) (see Figure 11).

(i) By Fact 8 H' is an expanded, acyclic digraph. Because of the existence of the edge (u, v), u is the only source of H'.

(ii) By Fact 8 H' is upward planar.

2. If u and v are incomparable in K, by Lemma 4 there exists in K a path P_K homeomorphic to a valley with $a \equiv u$ and $b \equiv v$ (see Figure 11). Thus, $H'' \equiv H \cup (s, v) \cup (s, u)$ is homeomorphic to a subgraph of G. Observe that in H'', vertex u has in-degree 1. If we contract the edge (s, u) we obtain $H''/(s, u) \equiv H \cup d_K = H'$.

(i) By Fact 8 H'' is acyclic and expanded; thus by Lemma 7 H' is acyclic. Note that all vertices except u have the same in- and out-degrees in H'' and H'. Since H'' is expanded and u is a source of H', then H' is an expanded digraph. Moreover, since $s(G) \in K^\circ$, u is the only source of H'.

(ii) By Fact 8 H'' is upward planar; thus, by Lemma 7 H' is upward planar.

3. If there exists a path P_K from v to u in K, then v is also a source of H (see Figure 11). Otherwise u would be the only source of H, so there would be a path $u \to v$ in H and thus a path $u \to v$ in G, which is a contradiction. So both u and v are sources of H.

(i) Since H is an expanded, acyclic digraph, and since both vertices u and v are sources of H, $H' = H \cup (u, v)$ is expanded and acyclic. Furthermore, it contains only one source u.

(ii) Let H'' be the digraph $H \cup (v, u)$. Digraph H'' is homeomorphic to a subgraph of G and thus, by Fact 8, there exists an upward embedding $\Psi_{H''}$ of H'', with (v, u) on the external face. Furthermore, v is the only source of H'', and H'' is an expanded sT-digraph.

H' can be obtained from H'' by the means of the following operations:

1. Construct $\bar{H}$ from H'' by adding a vertex s' and replacing the edge (v, u) with the valley $\{(s', v), (s', u)\}$.

2. Construct H' from $\bar{H}$ by contracting the edge (s', u).

By Lemma 8, $\bar{H}$ has an upward embedding with the valley $\{(s', v), (s', u)\}$ on the external face. By Lemma 7, H' has an upward embedding with the edge (u, v) on the external face. □

Following the developments of the previous proof, for each case a particular path P_K is detected in the component K. These paths and the corresponding cases will play a central role in the next section.

The previous lemma refers to a particular minor of G obtained by replacing exactly one component, but it can be easily extended to any minor of G. Before proving the next lemma, we must observe a property of sources in minors. Suppose u is a source of G. Then u is a source in every component of G that contains u. By a simple inspection of Rules 1–4 we have the following.

FACT 9. *Let G be a digraph and K be a component of G with respect to the separation pair $\{u, v\}$. Let G' be a minor of G such that $K \subset G'$. If u is a source of $G - K$, then u is a source in $G' - K$.*

LEMMA 10. *Let G be an expanded sT-digraph and let $\tilde{H}' = G - K_1 - \cdots - K_m \cup d_{K_1} \cdots \cup d_{K_m}$ be a minor of G, where $d_{K_i} = d(K_i, G - K_i)$. Then*

(i) *$\tilde{H}'$ is an expanded sT-digraph.*

(ii) *if G is upward planar, then $\tilde{H}'$ is upward planar.*

(iii) *if G is upward planar and $s(G) \in K_i^\circ$, $1 \le i \le m$, then H' has an upward embedding with d_{K_i} on the external face.*

Proof. The proof is by induction. When $m = 1$, the minor is $\tilde{H}' = G - K_1 \cup d(K_1, G - K_1)$, and by Lemma 9 the basis of the induction holds. Now, suppose that the minor $\tilde{G} = G - K_1 - \cdots - K_{l-1} \cup d_{K_1} \cup \cdots \cup d_{K_{l-1}}$ is an expanded sT-digraph and is upward planar. We show that the minor $\tilde{J}' = G - K_1 - \cdots - K_l \cup d_{K_1} \cup \cdots \cup d_{K_l}$ is an expanded sT-digraph and is upward planar.

Note first that $\tilde{J}' = \tilde{G} - K_l \cup d_{K_l}$. If we can prove that $d_{K_l} = d(K_l, G - K_l)$ is equal to $d(K_l, \tilde{G} - K_l)$, then the thesis follows by Lemma 9. In other words, we have to prove that the directed-virtual-edge which substitutes K_l remains the same when Rules 1–4 are applied to the pair $(K_l, \tilde{G} - K_l)$ rather than to the pair $(K_l, G - K_l)$.

When Rules 1 and 2 are applied, the directed-virtual-edge depends only on the component K_l, and $d_K = d(K_l, G - K_l) = d(K_l, \tilde{G} - K_l)$. Hence we have to consider only Rules 3 and 4.

3(a) u is a source of K_l and v is an internal vertex of K_l, v is a source of $G - K_l$ and $s(G) \notin K_l^\circ$.

By Fact 9, v is a source in $\tilde{G} - K_l$. Moreover, since $s(G) \notin K_l^\circ$ we have that $s(\tilde{G}) \notin K_l^\circ$, and Rule 3(a) must be applied to the pair $(K_l, \tilde{G} - K_l)$.

3(b) u is a source of K_l and v is an internal vertex of K_l; v is not a source of $G - K_l$ or $s(G) \in K_l^\circ$.

If $s \in K_l^\circ$, Rule 3(b) must be applied to the pair $(K_l, \tilde{G} - K_l)$.

Suppose that v is not a source of $G - K_l$ and $s \notin K_l^\circ$. Note that v is a sink in $G - K_l$ (v cannot be internal in $G - K_l$ since v is internal in K_l and G is expanded).

If v is not a source of $\tilde{G} - K_l$ then Rule 3(b) must be applied to the pair $(K_l, \tilde{G} - K_l)$.

Suppose v is a source of $\tilde{G} - K_l$; then there exists a component K_q of $G - K_l$, with $0 < q < l$, with $K_q \notin \tilde{G} - K_l$, having v and u_q as poles, whose directed-virtual-edge is either a peak or the edge (v, u_q). Observe that v is a sink of K_q. If $s \notin K_q^\circ$ then u_q is the only source of K_q. Hence Rule 2(a) is applied to the pair $(K_q, G - K_q)$ and $d(K_q, G - K_q)$ is the edge (u_q, v), which is a contradiction.

If $s \in K_q^\circ$ then u_q is a source of $G - K_q$ (since v is internal in $K_l \in G - K_q$). Thus u_q is not a source of K_q; otherwise G contains two sources. Then Rule 4(a) is applied to the pair $(K_q, G - K_q)$ and $d(K_q, G - K_q)$ is the edge (u_q, v), which is a contradiction.

4(a) u and v are not sources of K_l; u is a source of $G - K_l$. By Fact 9, u is a source of $\tilde{G} - K_l$ and Rule 4(a) can be applied to the pair $(K_l, \tilde{G} - K_l)$.

4(b) u and v are not sources of K_l; u is a not a source of $G - K_l$. Since u is not a source in $G - K_l$ and $s(G) \in K_l$, then v is a source in $G - K_l$ and v is a source in $\tilde{G} - K_l$. So Rule 4(a) can be applied to the pair $(K_l, \tilde{G} - K_l)$, with v and u interchanged. □

The proof of the necessity of Theorem 3 is now a simple corollary of Lemma 10. In fact, for each node μ of tree $\mathcal{T}$, the sT-skeleton of μ is a minor of G.

6. Proof of sufficiency for Theorem 3.

LEMMA 11. *Let G be a planar expanded sT-digraph, $\{u, v\}$ a separation pair of G, and K a component with respect to $\{u, v\}$ such that $s(G) \in K$. Let $H = G - K$ and let $d_K = d(K, H)$ and $d_H = d(H, K)$ be the directed-virtual-edges associated to K and H (with respect to G), respectively. Finally, let H' be the minor $H \cup d_K$ and K' be the minor $K \cup d_H$. If K' is upward planar and H' has an upward embedding with d_K on the external face, then G is upward planar.*

Before proving the above lemma, we need some preliminary results, namely, the following Lemmas 12–16.

We remind the reader that in the proof of Lemma 9, in correspondence with each case of the proof, a path P_K is detected in the component K. Such a path P_K will be used in the following lemma (see Figure 12).

LEMMA 12. *Let G be a planar expanded sT-digraph G, $\{u, v\}$ a separation pair of G, and K a component with respect to $\{u, v\}$. Let $H = G - K$ and let $d_K = d(K, H)$ be the directed-virtual-edge associated to K with respect to G. Let H' be the minor $H \cup d_K$ and let $\bar{H} = H \cup P_K$. Suppose H' has an upward embedding $\Psi_{H'}$, with d_K embedded on the external face if $s(G) \in K^\circ$. Denote by $\Psi_H \subset \Psi_{H'}$ the upward embedding of H contained in $\Psi_{H'}$ and let α_H be the face of Ψ_H in which d_K is embedded. We have that*

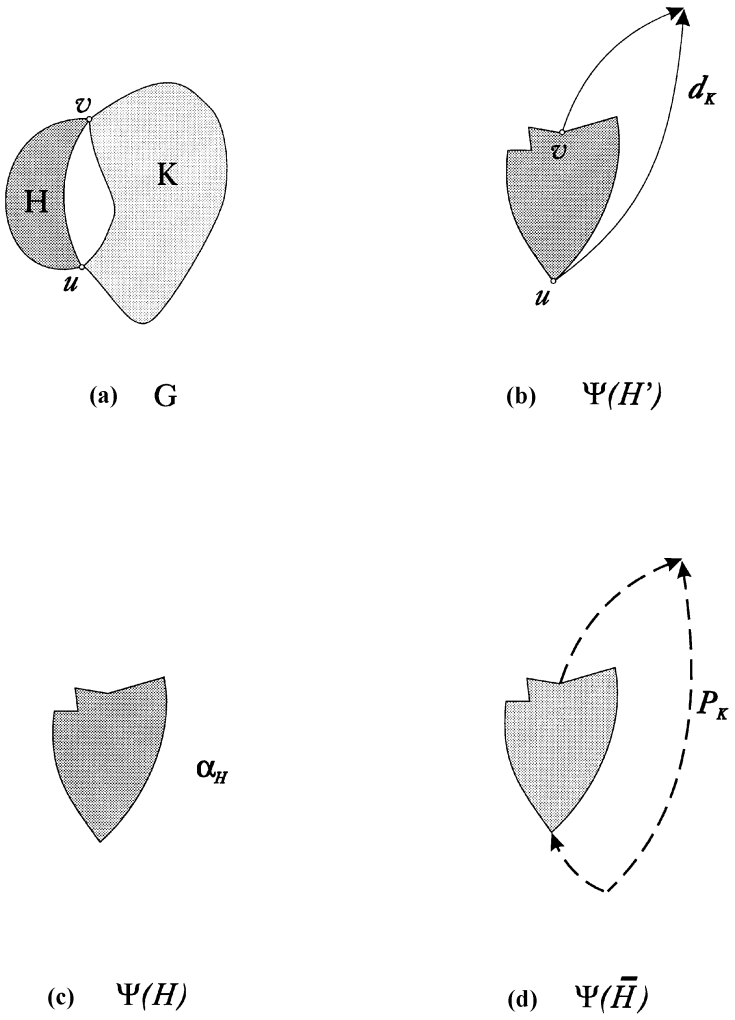

FIG. 12. *Illustration of the statement of Lemma* 12.

(i) $\bar{H} = H \cup P_K$ *is an expanded, acyclic digraph.*

(ii) $\bar{H}$ *is an sT-digraph and has an upward embedding* $\Psi_{\bar{H}}$*, with* $\Psi_H \subset \Psi_{\bar{H}}$ *and* P_K *embedded in face* α_H *of* Ψ_H*.*

Proof. Since $\bar{H}$ is a subgraph of G then it is expanded and acyclic and (i) holds.

We now prove (ii). Since both d_K and P_K depend on the component K, we distinguish the following four cases, each corresponding to the cases of the proof of Lemma 9.

1. P_K is homeomorphic to d_K; hence $\bar{H}$ is homeomorphic to H'. By Fact 8, the lemma holds.

2.(a) P_K is homeomorphic to d_K; hence $\bar{H}$ is homeomorphic to H'. By Fact 8, the lemma holds.

(b) Since $s \in K^\circ$, d_K is embedded on the external face of $\Psi_{H'}$. Observe that $\bar{H}$ can be obtained from H' substituting the edge (v, t) with a path homeomorphic to a valley. Thus, by Lemma 8 and by Fact 8, the lemma holds.

3.(a) P_K is homeomorphic to d_K; hence $\bar{H}$ is homeomorphic to H'. By Fact 8, the lemma holds.

(b) If $s \in K^\circ$ then the proof is as in case 2(b). If $s \notin K^\circ$ then P_K is homeomorphic to d_K; hence $\bar{H}$ is homeomorphic to H'. By Fact 8, the lemma holds.

4. (1) P_K is homeomorphic to d_K; hence $\bar{H}$ is homeomorphic to H'. By Fact 8, the lemma holds.

(2) Since $s \in K^\circ$, d_K is embedded on the external face of $\Psi_{H'}$. Observe that $\bar{H}$ can be obtained from H' substituting the edge (u, v) with a path homeomorphic to a valley. Thus, by Lemma 8 and by Fact 8, the lemma holds.

(3) Observe that $\bar{H}$ can be obtained from H' by reversing edge (u, v) and by direct subdivision. Thus, following the proof of case 4.3 of Lemma 9, and by Fact 8, the lemma holds.

Suppose K, H, K', and H' satisfy the conditions of Lemma 11. Let P_K and P_H be the paths associated with d_K and d_H, respectively (see Figures 13(a), (b)). Except for the common endpoints u and v, P_K and P_H are disjoint paths of G, since they lie in different components. Thus, $C = P_K \cup P_H$ is a simple (undirected) cycle of G.

Let $\bar{K} = K \cup P_H$ and $\bar{H} = H \cup P_K$ (see Figures 13(c), (d)). Since K' and H' are upward planar, by the previous lemma, $\bar{K}$ and $\bar{H}$ are upward planar. Let $\Psi_{\bar{K}}$ and $\Psi_{\bar{H}}$ be two upward embeddings of $\bar{K}$ and $\bar{H}$, respectively, and let $\alpha_{\bar{K}}$ and $\alpha_{\bar{H}}$ be the corresponding external faces. Now, let $K^*(H^*)$ be the subgraph of $\bar{K}(\bar{H})$ embedded inside C in $\Psi_{\bar{K}}(\Psi_{\bar{H}})$ (see Figures 13(e), (f)). We have the following lemma.

LEMMA 13. *Let s^* be a source of K^* (H^*). Then $s^* \in C$.*

Proof. Suppose $s^* \notin C$. Then s^* is a source of $\bar{K}$ embedded inside C in $\Psi_{\bar{K}}$. Since $\bar{K}$ is an sT-digraph, s^* is the only source of $\bar{K}$ and is embedded on the external face of every upward embedding of $\bar{K}$, and thus cannot be embedded inside C in $\Psi_{\bar{K}}$, which is a contradiction. □

LEMMA 14. *The digraph K^* (H^*) is an expanded sT-digraph.*

Proof. Since K^* is a subgraph of G, then it is acyclic and expanded. By Lemma 13, all of the sources of K^* belong to $C = P_K \cup P_H$. In order to prove the existence of a single source, we have to consider the "shapes" of P_K and P_H. Observe that P_K (P_H) can be homeomorphic to an edge, a peak, a valley, or a zig-zag. If both P_K and P_H are homeomorphic to the edge (u, v) $((v, u))$, then $s(K^*)$ is either u or v. Suppose now that P_K is not homeomorphic to an edge. We consider the following cases.

1. P_K is homeomorphic to a peak. Clearly, if P_H is homeomorphic to (u, v) or (v, u) or a valley, then C has only one source. We now show that P_H is not homeomorphic to a peak or to a zig-zag. Since P_K is a peak, then K is in case 1 or in case 3b.2 of the proof of Lemma 9.

Suppose P_H is a peak. Then H is in case 1 or in case 3b.2.

(a) P_K as in case 1. If H is in case 1, then G contains two sources, which is a contradiction. If H is in case 3b.2, then v is not a source of K, which is a contradiction.

(b) P_K as in case 3b.2. If H is in case 1, then the proof is as above. If H is in case 3b.2, then v is internal both in H and in K, which is a contradiction.

Suppose P_H is homeomorphic to a zig-zag. Then H is in case 2.b or in case 3b.1.

(a) P_K as in case 1. If H is in case 2.b or in case 3b.1, then $s(G) \in H^\circ$. Since u is source both in K and in H then G contains two sources, which is a contradiction.

(b) P_K as in case 3b.2. The proof is as above.

2. P_K homeomorphic to a zig-zag (w.l.o.g., we can suppose $a \equiv u$ and $b \equiv v$). Clearly, if P_H is homeomorphic to (v, u), then C has only one source. If P_H is

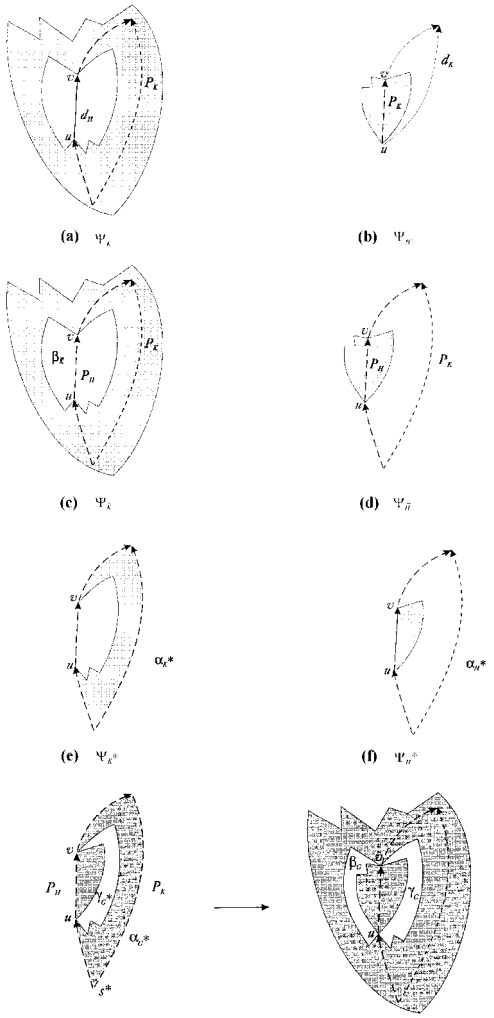

FIG. 13. *Construction of* Ψ_G.

homeomorphic to a peak, the proof is as for the case P_H homeomorphic to a zig-zag and P_K homeomorphic to a peak. We now show that P_H is not homeomorphic to the edge (u, v), to a valley, or to a zig-zag. Since P_K is a zig-zag, then K is in case 2(b) or case 3b.1; thus $s(G) \in K^\circ$ and u is a source of K.

Suppose P_H is homeomorphic to the edge (u, v). Then H is in case 2(a) or case 3(a) or case 4.1. If H is in case 2(a) or case 3(a), then u is a source of G and G contains two sources, which is a contradiction. If H is in case 4.1, then $s(G) \in H^\circ$, which is a contradiction.

Suppose P_H is homeomorphic to a valley. Then H is in case 4.2 and $s(G) \in H^\circ$, which is a contradiction.

Suppose P_H is homeomorphic to a zig-zag. Then H is in cases 2(b) and 3b.1 and $s(G) \in H^\circ$, which is a contradiction.

3. P_K is homeomorphic to a valley. Clearly, if P_H is homeomorphic to the edge (u, v), to the edge (v, u), or to the peak, then C has only one source. If P_H is

homeomorphic to a zig-zag, the proof is as for the case where P_H is homeomorphic to a valley and P_K is homeomorphic to a zig-zag. We now show that P_H is not homeomorphic to a valley.

Since both P_K and P_H are homeomorphic to a valley, then both K and H are in case 4.2; thus $s(G) \in K^\circ$ and $s(G) \in H^\circ$, which is a contradiction. □

The proof of the next lemma can be found in [3, 4].

LEMMA 15. *Let G be digraph, let Ψ_G be a candidate planar embedding of G, and let face $\delta \in \Psi_G$. Finally, let $\mathcal{A}$ be an assignment of the sinks and the sources of G to the faces of Ψ_G. If $|A(f)| = c(f)$ for each face $f \in \Psi_G$, with $f \neq \delta$, then $|A(\delta| = c(\delta)$.*

Suppose G, K, H, K', and H' satisfy the conditions of Lemma 11. In the following we give a constructive procedure to derive an embedding Ψ_G of G from two upward embeddings $\Psi_{H'}$ and $\Psi_{K'}$ of H' and K', respectively. We then show Ψ_G to be upward.

Let $\bar{K}$, $\bar{H}$, K^*, H^*, Ψ_{K^*}, Ψ_{H^*} be as defined for Lemmas 13 and 14 and let $\Psi_{K^*} \subseteq \Psi_{\bar{K}}(\Psi_{H^*} \subseteq \Psi_{\bar{H}})$ be the upward embedding of K^* (H^*) contained in $\Psi_{\bar{K}}$ ($\Psi_{\bar{H}}$). Denote by α_{K^*} and α_{H^*} the external faces of Ψ_{K^*} and Ψ_{H^*}, respectively (see Figures 13(e), (f)). Recall that $\alpha_{K^*} = \alpha_{H^*} = C$.

Let $G^* = K^* \cup H^*$ (see Figure 13(g)). Ψ_{G^*} is a planar embedding of G^* such that $\Psi_{K^*} \subset \Psi_{G^*}$, $\Psi_{H^*} \subset \Psi_{G^*}$, and P_K and P_H lie on the external face α_{G^*} of Ψ_{G^*} (i.e., $\alpha_{G^*} = C$).

We denote by γ_{G^*} the other face of Ψ_{G^*} (besides α_{G^*}) containing both edges of K^* and edges of H^*.

LEMMA 16. *Let G^* and Ψ_{G^*} be defined as above. Then*

(i) *G^* is an expanded sT-digraph.*

(ii) *Ψ_{G^*} is an upward embedding.*

Proof. (i) Since G^* is a subgraph of G, then G^* is expanded and acyclic. Since G^* is acyclic, it contains at least one source. Suppose G^* contains two sources s_1 and s_2. Since K^* (H^*) is an sT-digraph, both s_1 and s_2 cannot belong to K^* (H^*). Thus, w.l.o.g., $s_1 \in K^*$ and $s_2 \in H^*$. Furthermore, by Lemma 13, s_1 and s_2 lie on cycle C, so they are both contained in K^* and H^*, which is a contradiction. In the following, we denote by s^* the source of G^*.

(ii) We derive an upward consistent assignment $\mathcal{A}_{G^*}$ from the upward consistent assignments $\mathcal{A}_{K^*}$ and $\mathcal{A}_{H^*}$ associated with Ψ_{K^*} and Ψ_{H^*}.

First note that $A_{K^*}(\alpha_{K^*}) = A_{H^*}(\alpha_{H^*})$. In fact, let T^* be the set of sink-switches of $C = \alpha_{K^*} = \alpha_{H^*}$. Since Ψ_{K^*} and Ψ_{H^*} are upward embeddings of sT-digraphs, by Fact 2, each sink-switch on α_{K^*} and α_{H^*} is a sink of K^* and H^*, respectively, and they are all assigned to α_{H^*} and α_{K^*}. We have that $A_{K^*}(\alpha_{K^*}) = A_{H^*}(\alpha_{H^*}) = T^* \cup \{s^*\}$.

For each face $f \in \Psi_{G^*}$, with $f \neq \gamma_{G^*}$ and $f \neq \alpha_{G^*}$, we have that f belongs to Ψ_{K^*} or f belongs to Ψ_{H^*}, but not both. It is trivial to see that the following assignment to the faces of $\Psi_{G^*} - \{\gamma_{G^*}\}$ is feasible (i.e., the number of vertices assigned to each face equals the capacity of the face):

- $A_{G^*}(\alpha_{G^*}) = A_{K^*}(\alpha_{K^*}) = A_{H^*}(\alpha_{H^*})$;
- $A_{G^*}(f) = A_{K^*}(f)$, for $f \neq \alpha_{G^*}$ and $f \in \Psi_{K^*}$;
- $A_{G^*}(f) = A_{H^*}(f)$, for $f \neq \alpha_{G^*}$ and $f \in \Psi_{H^*}$.

Observe that all the sinks of Ψ_{G^*} not assigned by the above assignment lie on face γ_{G^*}, and so they can be assigned to it in $\mathcal{A}^*_G$. Since Ψ_{G^*} is a candidate embedding and, for each face $f \in \Psi_{G^*} - \{\gamma_{G^*}\}$, it is $|A_{G^*}(f)| = c(f)$, by Lemma 15 we have that $A_{G^*}(\gamma_{G^*}) = c(\gamma_{G^*})$. Thus, A_{G^*} is an upward consistent assignment and G^* is

upward planar. □

We are now able to give the proof of Lemma 11.

Proof of Lemma 11. A planar embedding Ψ_G of G can be obtained from the upward embeddings $\Psi_{\bar{K}}$ and $\Psi_{\bar{H}}$ in such a way that $\Psi_{G^*} \subseteq \Psi_G$. Each face of Ψ_G belongs either to $\Psi_{\bar{K}}$ or to $\Psi_{\bar{H}}$, except for two faces which share edges both of $\bar{K}$ and $\bar{H}$. One of these two faces coincides with face γ_{G^*} of Ψ_{G^*}, and thus it is internal in Ψ_{G^*}; we denote it by γ_G and denote the other face by β_G (see Figure 13(h)). Conversely, all the faces of $\Psi_{\bar{K}}$ ($\Psi_{\bar{H}}$), except for the two faces sharing P_H (P_K), belong to Ψ_G.

We now derive an assignment $\mathcal{A}_G$ from the upward consistent assignments $\mathcal{A}_{\bar{K}}$, $\mathcal{A}_{\bar{H}}$, and $\mathcal{A}_{G^*}$ in the following way:

- $A_G(\gamma_G) = A_{G^*}(\gamma_{G^*})$;
- $A_G(f) = A_{\bar{K}}(f)$, for $f \in \Psi_{\bar{K}}$;
- $A_G(f) = A_{\bar{H}}(f)$, for $f \in \Psi_{\bar{H}}$;
- it is easy to see that all remaining sinks and (eventually) the source of G stay on face β_G and so they are assigned to β_G in $\mathcal{A}_G$.

In order to prove that $\mathcal{A}_G$ is upward consistent we have to show that

(i) every sink and the source of G is assigned to exactly one face of Ψ_G;
(ii) every internal vertex of G is not assigned to any face of Ψ_G;
(iii) the number of vertices assigned to each face equals the capacity of the face.

Observe first that since P_K is embedded in the external face $\alpha_{\bar{H}}$ of $\Psi_{\bar{H}}$, $\alpha_{\bar{H}}$ is not a face of Ψ_G. Moreover, since γ_{G^*} is not the external face of Ψ_{G^*}, u and v are not assigned to it in $\mathcal{A}_{G^*}$ (they both lie on the external face) and, in turn, in $\mathcal{A}_G$.

We first prove (i). It is easy to see that each sink (source) of G is assigned to at least one face. We have to prove that each sink (source) of G is assigned to at most one face. Let x be a vertex assigned to two faces f_1 and f_2 in $\mathcal{A}_G$ ($f_1 \neq f_2$). From the definition of $\mathcal{A}_G$, $f_1 \in \Psi_{\bar{K}}$ and $f_2 \in \Psi_{\bar{H}}$. This is not possible if $x \neq u$ or $x \neq v$. If $x = u$ ($x = v$) then it is assigned to the external face $\alpha_{\bar{H}} = f_2$ of $\Psi_{\bar{H}}$ in $\mathcal{A}_{\bar{H}}$ and thus $f_2 \notin \Psi_G$, which is a contradiction.

(ii) Let x be an internal vertex of G and suppose it is assigned to a face f of Ψ_G. Again $x = u$ or $x = v$. Assume, w.l.o.g., $x = v$. Observe that f is not a face of $\Psi_{\bar{H}}$ (v is eventually assigned to its external face, which is not a face of Ψ_G). Furthermore, $f \neq \gamma_G$, since neither u nor v is assigned to it in $\mathcal{A}_G$. So f is a face of $\Psi_{\bar{K}}$.

Let us denote by $\beta_{\bar{K}}$ the face of $\Psi_{\bar{K}}$ sharing the path P_K and not embedded inside cycle C (see Figure 13(c)). Face $\beta_{\bar{K}}$ is not a face of Ψ_G, and thus $f \neq \beta_{\bar{K}}$. Since v is not assigned to $\beta_{\bar{K}}$ in $\mathcal{A}_{\bar{K}}$, $\beta_{\bar{K}}$ is internal in $\Psi_{\bar{K}}$. So, both $\beta_{\bar{K}}$ and $\gamma_{\bar{K}}$ are internal faces of $\Psi_{\bar{K}}$ and hence Ψ_H is embedded in a face of Ψ_K in Ψ_G. This implies that $s(\bar{K}) = s(G) \in \bar{K}$.

Suppose now that $x = v$ is a sink in $\bar{K}$ and v is not a sink in G.

Since v is a sink of $\bar{K}$, then P_H has an incoming edge into v, and P_H is homeomorphic to the edge (u, v) or to a zig-zag or to a valley. If P_H is homeomorphic to edge (u, v), then H is in case 2(a), 3(a), or 4.1 of the proof of Lemma 9. In case 2(a), v is a sink of H and thus u is a sink of G, which is a contradiction. In case 3(a), v is a source of K and thus it is not a sink in $\bar{K}$, which is a contradiction. If H is in case 4.1 then $s \in H^\circ$, which is a contradiction.

If P_H is homeomorphic to a valley, then H is in case 4.2 and $s(G) \in H^\circ$, which is a contradiction.

If P_H is homeomorphic to a zig-zag, then H is in case 2(b) or 3b.2. In both cases, $s(G) \in H^\circ$, which is a contradiction.

(iii) Since G is an expanded digraph, every planar embedding is candidate. By Lemma 15, the capacity equation for face β_G is satisfied, and $\mathcal{A}_G$ is upward consistent. □

Lemma 11 refers to digraph G. We extend the result to a minor $\tilde{G}$ of G. This is done by showing that, under certain restrictions, the directed-virtual-edges substituting components can be chosen independently one from another. In particular, this is true if the source s of G belongs to its minor. Note that in this case a minor of a minor of G is a minor of G.

LEMMA 17. *Let $\tilde{G}$ be a minor of G such that $s(\tilde{G}) = s(G)$. Let $\{u, v\}$ be a split pair of $\tilde{G}$ such that $\{u, v\}$ is also a split pair of G. Let $\tilde{K}$ be a component of $\tilde{G}$ w.r.t. $\{u, v\}$ and let K be the corresponding component of G, i.e., K is obtained from $\tilde{K}$ by replacing each directed-virtual-edge of $\tilde{K}$ with its associated component of G. Then it is $d(\tilde{K}, \tilde{G} - \tilde{K}) = d(K, G - K)$.*

Proof. In the following we denote by H the digraph $G - K$ and by $\tilde{H}$ the digraph $\tilde{G} - \tilde{K}$. Observe that, for each component J of G such that $J \notin \tilde{G}$ (i.e., J is substituted by its directed-virtual-edge), since $s(G) \in \tilde{G}$, then $s(G) \notin J^\circ$.

We examine the following four cases corresponding to the substitution Rules 1–4 applied to components K and $\tilde{K}$.

1. Since u and v are sources of K, by Fact 9, u and v are sources of $\tilde{K}$. Then $d(\tilde{K}, \tilde{H})$ is a peak and the lemma holds.

2(a) u is a source of K, v is a sink of K, and $s(G) \notin K^\circ$. By Fact 9, u is a source of $\tilde{K}$. Since $s(\tilde{G}) = s(G)$ then $s(G) \notin K^\circ$. We now show that v is a sink of $\tilde{K}$. In fact, v is a sink of all the components having v as a pole. By the substitution rules, whenever a component $J \in K$ has v as a sink then the associated directed-virtual-edge has an edge incoming into v except for Rules 2(b) and 4(b). But, in both cases, $s(G) \in J^\circ$, contradicting that $s \notin K_\circ$. Then Rule 2(a) is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is the edge (u, v).

2(b) u is a source of K, v is a sink of K, and $s(G) \in K^\circ$. By Fact 9, u is a source of $\tilde{K}$. Furthermore, $s(\tilde{G}) \in \tilde{K}^\circ$.

If v is a sink of $\tilde{K}$, then Rule 2(b) is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is a peak.

If v is internal of $\tilde{K}$, then Rule 3(b) is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is a peak.

If v is a source of $\tilde{K}$, then $\tilde{K}$ has three sources and the minor $\tilde{G}$ is not an sT-digraph (it contains at least two sources), which is a contradiction.

3(a) u is a source of K, v is internal of K, $s(G) \in H$, and v is a source of H. By Fact 9, u is a source of $\tilde{K}$ and v is a source of $\tilde{H}$.

If v is internal in $\tilde{K}$, then Rule 3(a) is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is the edge (u, v).

If v is a sink of $\tilde{K}$, then Rule 2(a) is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is the edge (u, v).

If v is a source of $\tilde{K}$, then v is a source of $\tilde{G}$. Since $s(\tilde{G}) = s(G) \neq v$, then $\tilde{G}$ has two sources, which is a contradiction.

3(b) u is a source of K and v is internal in K, $s(G) \in K^\circ$ or v is not a source of H.

By Fact 9, u is a source of $\tilde{K}$. Two cases are possible.

(i) $s(G) \in K^\circ$. Then v is the only source of H. In fact, if u is a source of H, then u is a source of G, which is a contradiction. By Fact 9, v is a source of $\tilde{H}$.

If v is internal in $\tilde{K}$, then Rule 3(b) is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is a peak.

If v is a sink of $\tilde{K}$, then Rule 2(b) is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is a peak.

If v is a source of $\tilde{K}$, then v is a source of $\tilde{G}$. Since $s(\tilde{G}) = s(G) \neq v$, then $\tilde{G}$ has two sources, which is a contradiction.

(ii) v is not a source of H and $s(G) \in H$. Since G is expanded, v is a sink of H.

If v is a source of $\tilde{K}$, then Rule 1 is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is a peak.

If v is internal in $\tilde{K}$ and v is not a source of $\tilde{H}$, then Rule 3(b) is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is a peak. If v is a source of $\tilde{H}$, there exists a component J of H, with $J \notin \tilde{H}$, having v and u_J as poles, whose directed-virtual-edge is either a peak or the edge (v, u_J). Observe that v is a sink of J and u_J is the source of J (since $s(G) \notin J^\circ$). But in this case, Rule 2(a) is applied to J and $d(J, G - J)$ is the edge (u_J, v), which is a contradiction.

If v is a sink of $\tilde{K}$, since v is internal in K, there exists a component J, with $J \in K$ and $J \notin \tilde{K}$, having v and u_J as poles, whose directed-virtual-edge is the edge (u_J, v). Furthermore, v is an internal vertex or a source of J. If v is a source of J then $d(J, G - J)$ is not the edge (u_J, v). If v is internal in J, then Rules 3(a) or 4(b) must be applied in order to have $d(J, G - J) = (u_J, v)$. Rule 3(a) implies that v is a source of $G - J$; hence v is a source of K, which is a contradiction. Rule 4(a) implies that $s(G) \in J^\circ$, which is a contradiction.

4. u and v are not sources of K. $s \in K^\circ$. Either u or v (or both) is a source of H. Suppose, w.l.o.g., that u is a source of H. u is not a source of $\tilde{K}$; otherwise the minor $\tilde{G}$ has two sources. By Fact 9, u is a source of $\tilde{H}$.

If v is not a source of $\tilde{K}$, Rule 4(a) can be applied and $d(\tilde{K}, \tilde{H})$ is the edge (u, v).

If v is a source of $\tilde{K}$, there exists a component J of K, with $J \notin \tilde{K}$, having v and u_J as poles, such that v is a nonsource of J, and its directed-virtual-edge is either a peak or the edge (v, u_J). Observe that, since $s(G) \notin J^\circ$, u_J is the source of J. If v is a sink of J, Rule 2(a) is applied to J and $d(J, G - J)$ is the edge (u_J, v), which is a contradiction. If v is internal in J, Rule 3(b) is applied and, since $s(G) \notin J^\circ$, v is a sink of $G - J$ and hence it is a sink in all components of $K - J$. Now, v is a source in the digraph $\tilde{K} - d(J, G - J)$. Then there exists a component Z of K, with $Z \neq J$ and $Z \notin \tilde{K}$, having v and u_Z as poles, such that v is a source in $d(Z, G - Z)$. Since v is a sink of Z and $s(G) \notin Z^\circ$ (since $s(G) \in \tilde{K}$), u_Z is a source of Z and then Rule 2(a) is applied. But then $d(Z, G - Z)$ is the edge (u_Z, v), which is a contradiction. □

We are now able to prove the sufficiency part of Theorem 3.

Proof of sufficiency of Theorem 3. Let $\mathcal{T}$ be the rooted *SPQR*-tree associated with the graph G, and let $\mu_1, \ldots, \mu_m$ be the sequence of nodes of $\mathcal{T}$ deriving from a depth-first-search (DFS) visit of $\mathcal{T}$, starting at its root. Let $Skel(\mu_i), i = 1, \ldots, m$ be the sT-skeleton associated with $\mu, \ldots, \mu_m$.

For each node μ_i, let d_i^c be the directed virtual edge of μ_i in the skeleton associated with the parent of μ_i, and let d_i^p be the directed-virtual-edge of the parent of μ_i in $Skel(\mu_i)$. Clearly, if $\mu_i = \mu_1$ is the root, then $d_i^p = d_1^p = \{\emptyset\}$. Finally, let $\tilde{G}_i = Skel(\mu_1) - d_2^c \cup (Skel(\mu_k) - d_2^p) - d_3^c \cup (Skel(\mu_3) - d_3^p) - \cdots - d_i^c \cup (Skel(\mu_i) - d_i^p)$.

We show that $\tilde{G}_i$ is a minor of G, with $s(G) \in \tilde{G}_i$, and that $\tilde{G}_i$ is upward planar.

For $i = 1$ we have that $\tilde{G}_1 = Skel(\mu_1)$ and the claim trivially holds. Suppose that $\tilde{G}_{l-1}$ is a minor of G with $s(G) \in \tilde{G}_{l-1}$, and that $\tilde{G}_{l-1}$ is upward planar. We show that $\tilde{G}_l$ is a minor of G, with $s(G) \in \tilde{G}_l$, and that $\tilde{G}_l$ is upward planar.

Let H be the pertinent graph of d_l^c and K be the pertinent graph of d_l^p. Recall that $K \cup H = G$ and that K and H share exactly two vertices. Moreover $d_l^p = d(K, H)$ and $d_l^c = d(H, K)$.

Let $J_1, \ldots, J_q$ be components of G contained in K, such that

$$\begin{aligned}\tilde{G}_{l-1} &= G - H - J_1 - \cdots - J_q \cup d(H, K) \cup d(J_1, G - J_1) \cup \cdots \cup d(J_q, G - J_q) \\ &= K - J_1 - \cdots - J_q \cup d(H, K) \cup d(J_1, G - J_1) \cup \cdots \cup d(J_q, G - J_q),\end{aligned}$$

and let $Z_1, \ldots, Z_r$ be split components of G contained in H, such that

$$\begin{aligned} \mathit{Skel}(\mu_l) &= G - K - Z_1 - \cdots - Z_r \cup d(K,H) \cup d(Z_1, G - Z_1) \cup \cdots \cup d(Z_r, G - Z_r) \\ &= H - Z_1 - \cdots - Z_q \cup d(K,H) \cup d(Z_1, G - Z_1) \cup \cdots \cup d(Z_r, G - Z_r). \end{aligned}$$

The digraph $\tilde{G}_l = K \cup H - J_1 - \cdots - J_q - Z_1 - \cdots - Z_q \cup d(J_1, G - J_1) \cup \cdots \cup d(J_q, G - J_q) \cup d(Z_1, G - Z_1) \cup \cdots \cup d(Z_r, G - Z_r)$ is a minor of G. In fact, since $J_i \in K$, $i = 1, \ldots, q$, and $Z_t \in H$, $t = 1, \ldots, r$, it follows that J_i and Z_t do not share any edge, for $i = 1, \ldots, q$ and $t = 1, \ldots, r$.

Since $s(G) \in \tilde{G}_{l-1}$, then $s(G) \in \tilde{G}_l$.

Let $\tilde{K} = \tilde{G}_{l-1} - d_l^c$ and $\tilde{H} = \mathit{Skel}(\mu_l) - d_l^p$. Clearly $\tilde{G}_l = \tilde{K} \cup \tilde{H}$. By Lemma 17, $d(\tilde{K}, \tilde{H}) = d(K,H) = d_l^p$ and $d(\tilde{H}, \tilde{K}) = d(H,K) = d_l^c$. Since $\tilde{K} \cup d(\tilde{H}, \tilde{K}) = \tilde{G}_{l-1}$ is upward planar and $\tilde{H} \cup d(\tilde{K}, \tilde{H}) = \mathit{Skel}(\mu_l)$ is upward planar with $d(\tilde{K}, \tilde{H}) = d_l^p$ embedded on the external face, then by Lemma 11, $\tilde{G}_l$ is upward planar.

We now show that $\tilde{G}_m = G$. By induction $\tilde{G}_m$ is a minor of G. Suppose $\tilde{G}_m \neq G$; then there exists a component J of G such that $J \notin \tilde{G}_m$ and $d(J, G - J) \in \tilde{G}_m$. Let μ_j be the node of $\mathcal{T}$ such that $d(J, G - J) \in \mathit{Skel}(\mu_j)$. $d(J, G - J)$ is associated with either the parent of μ_j or one of the children of μ_j. Since the tree $\mathcal{T}$ has been entirely visited then $d(J, G - J)$ has been substituted, and thus $d(J, G - J) \notin \tilde{G}_m$, which is a contradiction. □

7. Algorithm for general single-source digraphs. Let G be a biconnected single-source digraph. In this section we present an algorithm for testing whether G is upward planar.

ALGORITHM. *Test.*

1. Construct the expansion G' of G.

2. Test whether G' is planar. If G' is not planar, then return "not-upward-planar" and stop; else, construct an embedding for G'.

3. Test whether G' is acyclic. If G' is not acyclic, then return "not-upward-planar" and stop.

4. Construct the SPQR-tree $\mathcal{T}$ of G' and the skeletons of its nodes.

5. For each virtual edge e of a skeleton, classify each endpoint of e as a source, sink, or internal vertex in the pertinent digraph of e. Also, determine if the pertinent digraph of e contains the source.

6. For each node μ of $\mathcal{T}$, compute the sT-skeleton of μ.

7. For each R-node μ of $\mathcal{T}$

(a) test whether the sT-skeleton of μ is upward planar by means of algorithm *Embedded-Test*. If *Embedded-Test* returns "not-upward-planar," then return "not-upward-planar" and stop.

(b) mark the virtual edges of the skeleton of μ whose endpoints are on the external face in some upward drawing of the sT-skeleton of μ.

(c) for each unmarked virtual edge e of the skeleton of μ, constrain the tree edge of $\mathcal{T}$ associated with e to be directed towards μ.

(d) if the source is not in $\mathit{skeleton}(\mu)$, let ν be the node neighbor of μ whose pertinent digraph contains the source, and constrain the tree edge (μ, ν) to be directed towards ν.

8. Determine whether $\mathcal{T}$ can be rooted at a Q-node in such a way that orienting edges from children to parents satisfies the constraints of steps 7(c)–(d). If such a rooting exists then return "upward-planar"; else return "not-upward-planar."

For single-source digraphs that are not biconnected we apply the above algorithm to each biconnected component.

THEOREM 4. *Upward planarity testing of a single-source digraph with n vertices can be done in $O(n)$ time using $O(n)$ space.*

Proof. Steps 1 and 3 can be trivially performed in $O(n)$ time. Planarity testing in step 2 can also be done in $O(n)$ time [18]. The construction of the SPQR-tree and the skeletons of its nodes (step 4) takes time $O(n)$ using a variation of the algorithm of [17]. The preprocessing of step 5 consists essentially of a visit of $\mathcal{T}$ and can be done in $O(n)$ time. Let n_μ be the number of vertices of the skeleton of μ. The information collected in step 5 allows us to perform step 6 in $O(n)$ time and step 7(d) in $O(n_\mu)$ time. By Theorem 2, step 7(a) takes $O(n_\mu)$ time. The output of step 7(a) allows us to perform steps 7(b)–(c) in $O(n_\mu)$ time. Since $\sum_\mu n_\mu = O(n)$, the total complexity of step 7 is $O(n)$. Finally, step 8 consists of a visit of $\mathcal{T}$ and takes $O(n)$ time. □

To parallelize Algorithm *Test*, we need an efficient way of testing in parallel whether a planar single-source digraph with n vertices is acyclic. For this purpose, we can use the algorithm of [20], which runs in $O(\log^3 n)$ time on a CRCW PRAM with n processors. However, the particular structure of planar single-source digraphs allows us to perform this test optimally. The following characterization is inspired by some ideas in [20].

Let G be an embedded, expanded, planar single-source digraph. The *clockwise subgraph* of G is obtained by taking the first incoming edge of each internal vertex, in clockwise order. The *counterclockwise subgraph* of G is similarly obtained by taking the first incoming edge of each internal vertex, in counterclockwise order. Such subgraphs of G have all vertices with in-degree 1 or 0.

THEOREM 5. *An embedded, expanded single-source digraph G is acyclic if and only if both the clockwise and counterclockwise subgraphs of G are acyclic.*

Proof. The only-if part is trivial. For the if part, assume for contradiction that G is not acyclic, and consider an arbitrary drawing of G with the prescribed embedding and with the source on the external face. We will show the existence of a cycle in either the clockwise or counterclockwise subgraph. Let γ be a cycle of G that does not enclose any other cycle. Since the source of G must be outside γ, all the edges incident on vertices of γ and inside γ must be outgoing edges. Hence, γ is contained in the clockwise or counterclockwise subgraph depending on whether it is a clockwise or counterclockwise cycle. □

The structure of each connected component of the clockwise and counterclockwise subgraphs is either a source tree, or a collection of source trees with their roots connected in a directed cycle. Hence, one can test whether such subgraphs are acyclic using standard parallel techniques. Since expansion preserves acyclicity, we have the following theorem.

THEOREM 6. *Given an embedded planar single-source digraph G with n vertices, one can test if G is acyclic in $O(\log n)$ time with $n/\log n$ processors on an EREW PRAM.*

By applying the result of Theorem 6 and various parallel techniques (in particular [14, 28, 27]) we can efficiently parallelize algorithm *Test*.

THEOREM 7. *Upward planarity testing of a single-source digraph with n vertices can be done in $O(\log n)$ time on a CRCW PRAM with $n \log\log n/\log n$ processors using $O(n)$ space.*

As a consequence of Theorems 1 and 3, algorithm *Test* can be easily extended such that if the n-vertex digraph G is found to be upward planar, a planar st-digraph

G' with $O(n)$ vertices is constructed that contains G as a subdigraph. Hence, by applying the planar polyline upward drawing algorithm of Di Battista, Tamassia, and Tollis [12] to G' and then removing the vertices and edges of G' that are not in G, we obtain a planar polyline upward drawing of G.

THEOREM 8. *Algorithm* Test *can be extended so that it constructs a planar polyline upward drawing if the digraph is upward planar. The complexity bounds stay unchanged.*

7.1. Examples of application of algorithm *Test*. In this subsection we use two examples to illustrate the behavior of algorithm *Test* in performing the upward planarity testing. In the first example, the algorithm is applied to a graph which is not upward drawable; in the second example, the algorithm is applied to an upward drawable graph.

Example 1. In this example we consider the graph G of Figure 2a. We apply algorithm *Test* step by step.

1. The expansion graph G' of G is shown in Figure 14.

2. By a simple inspection it is possible to verify that G' is planar.

3. Again, by a simple inspection, it is possible to verify that G' is acyclic.

4. The SPQR-tree $\mathcal{T}$ of G' is shown in Figure 15. The skeletons of the nodes of $\mathcal{T}$ are shown in Figure 16 (the skeletons of the Q nodes are omitted).

5. Consider, for example, the virtual edge (2, 15) in skeleton μ_1 of Figure 16, which is the virtual edge of μ_4. The pertinent graph of μ_4 is subgraph G'_4 of G' induced by the node set $V(G') - \{13, 14, 24, 25\}$. By simple inspection, it is possible to verify that node 2 is internal in G'_4, while node 15 is a sink in G'_4. In addition, the source 1 of G' is contained in G'_4. In the same way, all other endnodes of virtual edges can be classified.

6. The sT-skeletons corresponding to the skeletons of Figure 16 are shown in Figure 17. Consider, for example, the sT-skeleton of μ_2. The pertinent graph G'_1 associated with the virtual edge (2, 13) of the skeleton of μ_2 (Figure 16) is the subgraph of G' induced by the node set $V(G') - \{24\}$. Using the classification performed in the preceding step, it is easy to verify that: (i) 2 is internal in G'_1 and 13 is a source in G'_1. In addition, the source 1 of G' belongs to G'°_1, hence, by Rule 3b the directed-virtual-edge associated with G'_1 is a peak. The pertinent graph associated with the virtual edge (2, 24) of μ_2 is the directed edge (2, 24). If is easy to see that the directed-virtual-edge associated with an edge u, v, is the edge (u, v) (Rule 3a). Thus, the directed-virtual-edge associated with the directed edge 2, 24 is again (2, 24). The same holds for edge (24, 13).

7. Now we perform Steps 7(a)–7(d) on all the R-nodes of $\mathcal{T}$.

(a) In Figure 18 we show the face-sink graphs associated with the sT-skeletons of the R-nodes μ_1, μ_5, and μ_{12} (the vertices associated with the faces are represented by squares). It is easy to verify that they all satisfy the conditions of Theorem 1; thus the algorithm *Embedded Test* will return "upward-planar" for every R-node. It also returns, for each R-node, the set of faces that can be external faces in an upward drawing of the associated sT-skeleton. In the figure, the nodes associated with these faces are indicated by black squares.

(b) By inspection of Figure 18, the unmarked edges are the following:

- μ_1: $\{(13, 14)\}$.
- μ_5: $\{(2, 5), (5, 8), (5, 11), (6, 8), (7, 8), (9, 11), (10, 11)\}$.
- μ_{12}: $\{(1, 3), (2, 3), (3, 4)\}$.

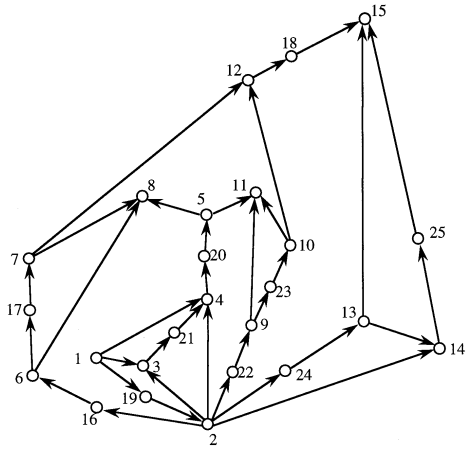

FIG. 14. *Expansion graph of graph G of Figure* 2a.

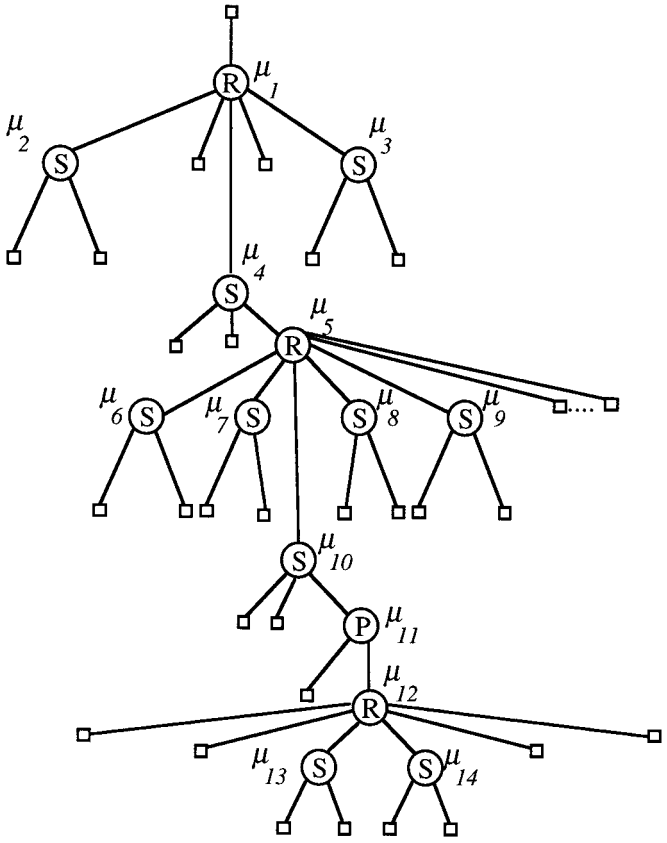

FIG. 15. *The SPQR-tree* $\mathcal{T}$ *of* G'.

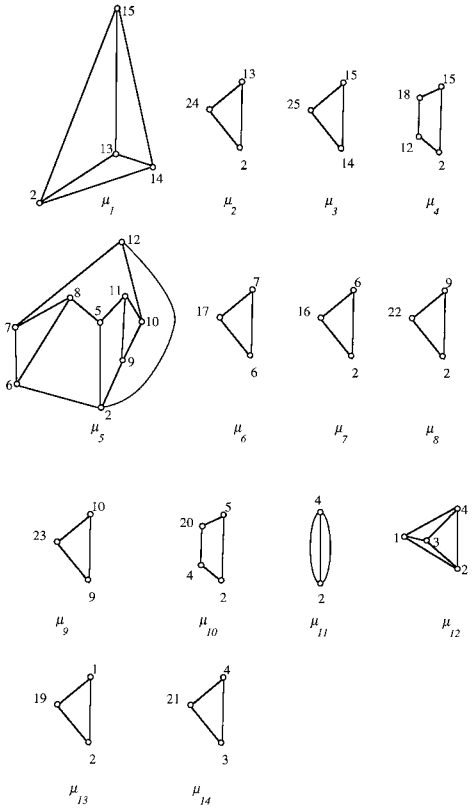

FIG. 16. *The skeletons of the nodes of* $\mathcal{T}$.

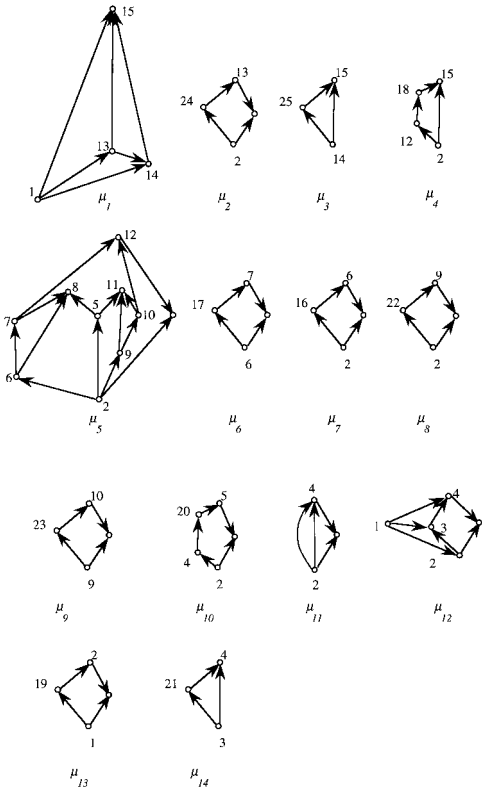

FIG. 17. *The sT-skeletons of the nodes of* $\mathcal{T}$.

(c) In Figure 19, the edges of the SPQR-tree $\mathcal{T}$ associated with unmarked virtual edges are oriented. For example, the tree-edge associated with the directed virtual edge (13, 14) of the sT-skeleton of μ_1 is oriented toward μ_1. Analogously, the tree-edge (μ_5, μ_{10}) associated with the directed-virtual-edge (2, 5) of sT-skeleton of μ_5 is directed toward μ_5.

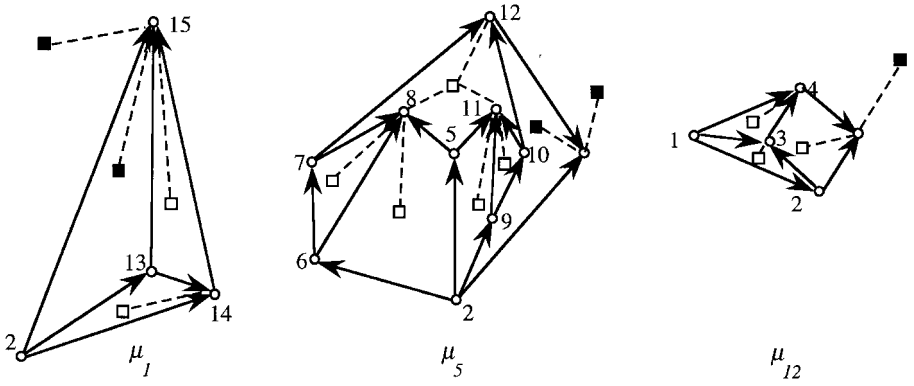

FIG. 18. *The face-sink graphs of the sT-skeletons of the* R*-nodes of* $\mathcal{T}$.

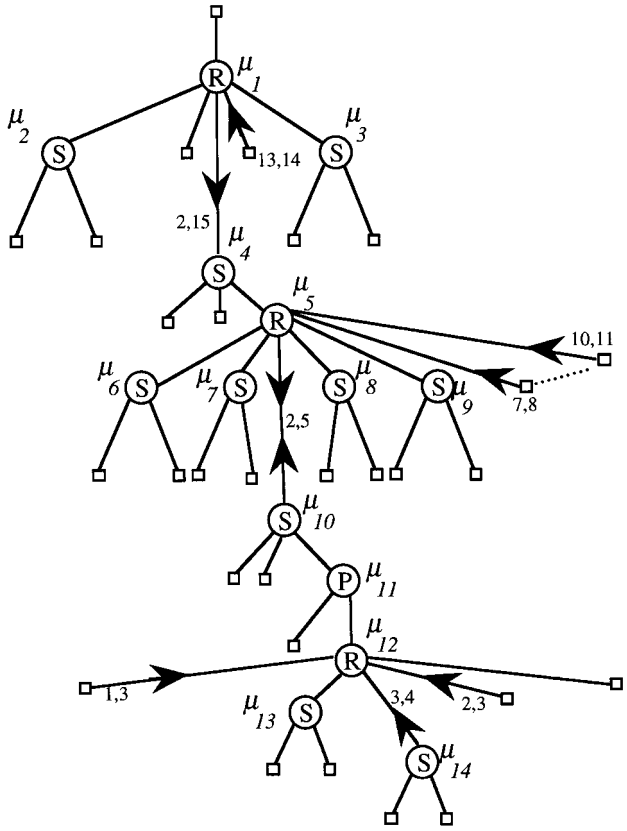

FIG. 19. *Partial orientation of* $\mathcal{T}$.

(d) Since the skeleton of μ_1 does not contain the source 1 of G', and the pertinent graph G'_4 associated with μ_4 contains the source 1, then the tree-edge μ_1, μ_4 is oriented toward μ_4. Analogously, the tree-edge μ_5, μ_{10} is oriented towards μ_{10}.

8. Since the tree-edge (μ_5, μ_{10}) is constrained to be oriented in both directions, it follows that it is not possible to find a rooting of $\mathcal{T}$ at a Q-node containing the source. Hence algorithm *Test* returns "not-upward-planar."

Example 2. In this example, we consider a graph H obtained from graph G of Figure 2a by contracting edge (12, 15).

1. In Figure 20 we show the expansion graph H' of H.

2. By a simple inspection it is possible to verify that H' is planar.

3. By a simple inspection it is possible to verify that H' is acyclic.

4. In Figure 21 we show the SPQR-tree $\mathcal{T}_H$ of H' and in Figure 22 the skeletons of the nodes of $\mathcal{T}_H$ (the skeletons of the Q-nodes are omitted).

5. In the same way as in step 5 of the previous example, all the endnodes of the virtual edges can be classified.

6. The sT-skeletons corresponding to the skeletons of Figure 22 are shown in Figure 23.

7. (a) In Figure 24 we show the face-sink graphs associated with the sT-skeletons of the R-nodes μ_1, μ_4, and μ_{11} (the vertices associated with the faces are represented by squares). It is easy to verify that they all satisfy the conditions of Theorem 1;

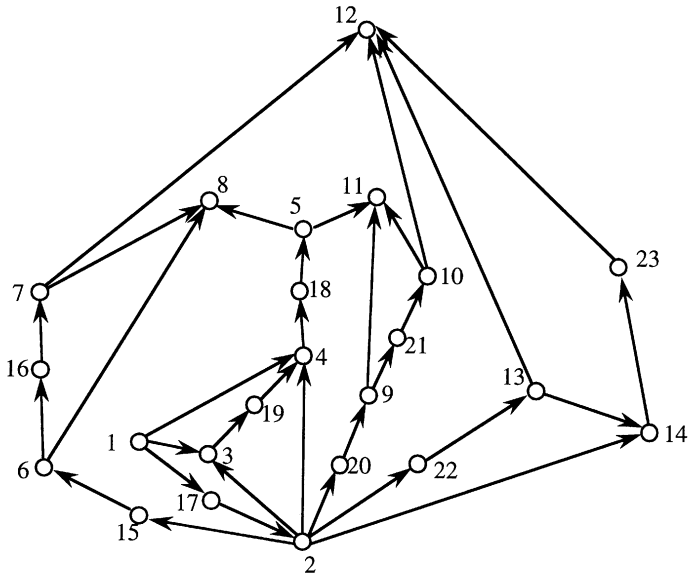

FIG. 20. *Expansion graph of graph H.*

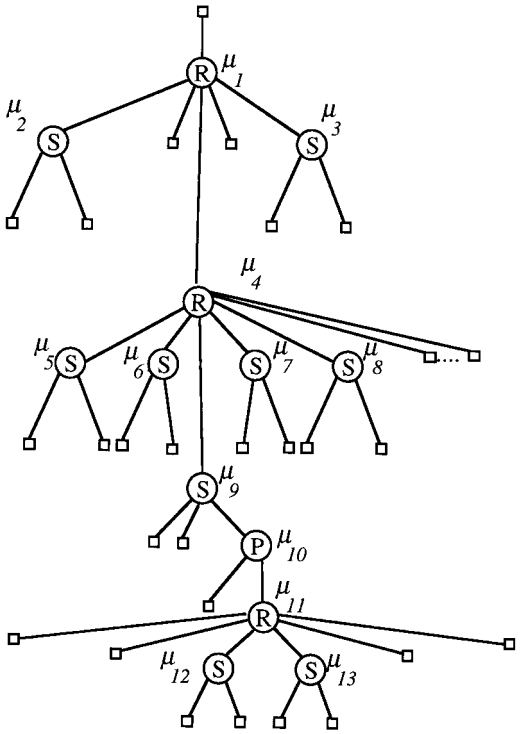

FIG. 21. *The SPQR-tree $\mathcal{T}_H$ of H'.*

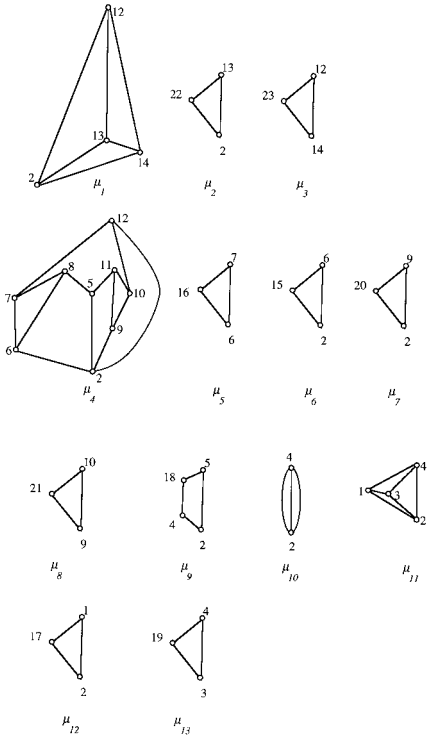

FIG. 22. *The skeletons of the nodes of* $\mathcal{T}_H$.

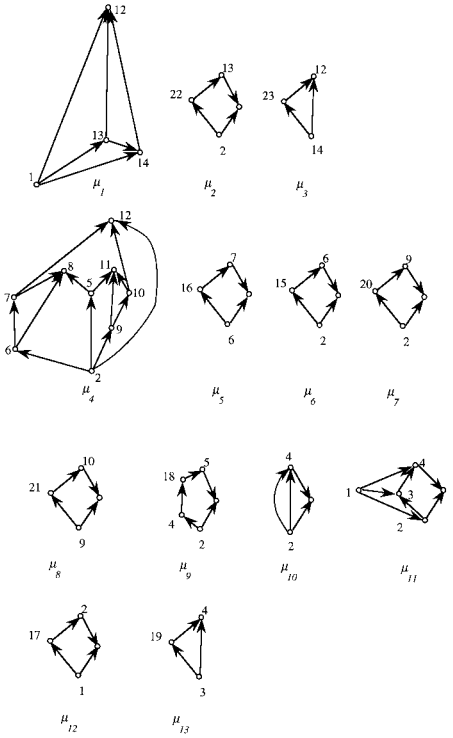

FIG. 23. *The* sT*-skeletons of the nodes of* $\mathcal{T}_H$.

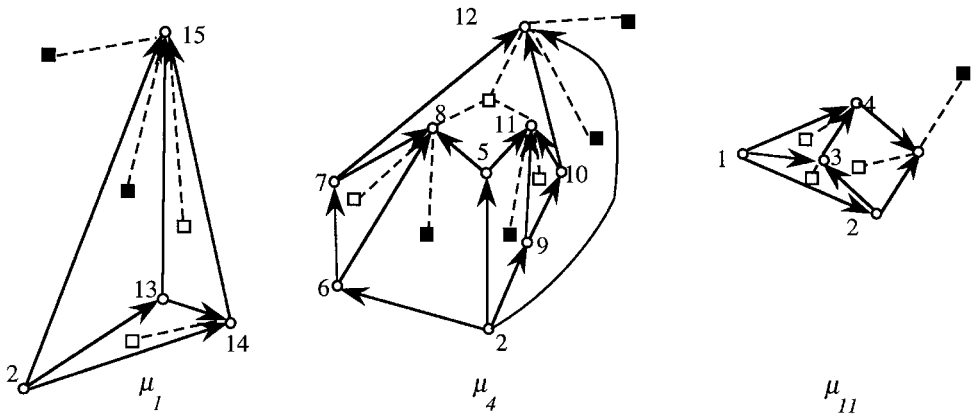

FIG. 24. *The face-sink graphs of the sT-skeletons of the* R*-nodes of* $\mathcal{T}_H$.

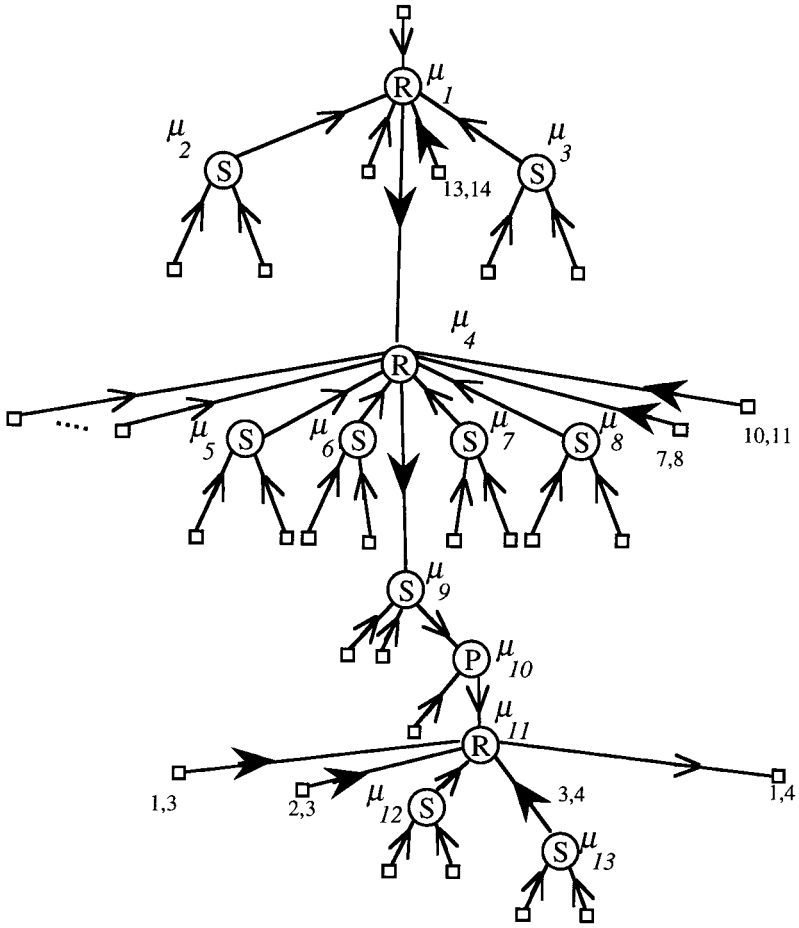

FIG. 25. *Orientation of* $\mathcal{T}_H$.

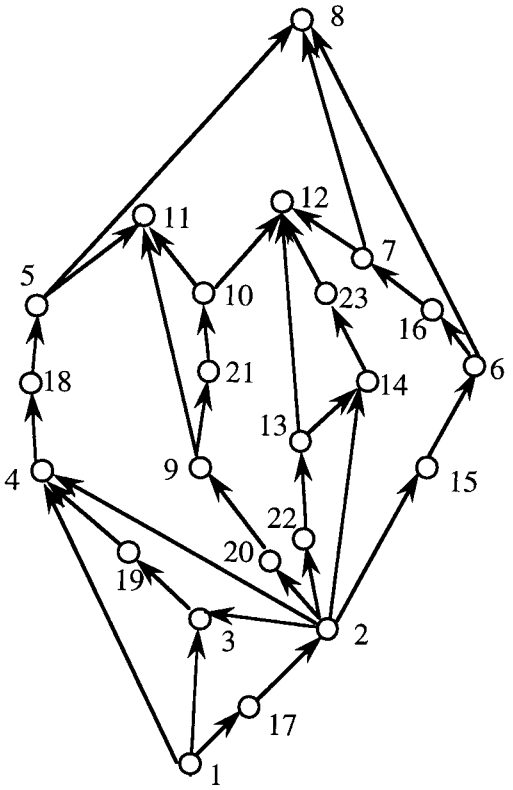

FIG. 26. *Upward drawing of H'.*

thus the algorithm *Embedded Test* will return "upward-planar" for every R-node. It also returns, for each R-node, the set of faces that can be external faces in an upward drawing of the associated sT-skeleton. In the figure, the nodes associated with these faces are indicated by black squares.

(b) By inspection of Figure 24, the unmarked edges are the following:

- μ_1: $\{(13, 14)\}$.
- μ_4: $\{(7, 8), (10, 11)\}$.
- μ_{11}: $\{(1, 3), (2, 3), (3, 4)\}$.

(c) In Figure 25, the edges of the SPQR-tree $\mathcal{T}_H$ associated with unmarked virtual edges are oriented; their orientations are represented by bold arrows.

(d) Since the skeleton of μ_1 does not contain the source 1 of H', and the pertinent graph H'_4 associated with μ_4 contains the source 1, then the tree-edge (μ_1, μ_4) is oriented toward μ_4. Analogously, the tree-edge (μ_4, μ_9) is oriented toward μ_9. Again, in Figure 25 these fixed orientations are represented by bold arrows.

8. By simple inspection, it is possible to find a rooting of $\mathcal{T}_H$ at the Q-node associated with the edge (1, 4) (see Figure 25). Hence algorithm *Test* returns "upward-planar."

An upward drawing of the expansion H' of graph H is shown in Figure 26.

REFERENCES

[1] P. Bertolazzi, R. F. Cohen, G. Di Battista, R. Tamassia, and I. G. Tollis, *How to draw a series-parallel digraph*, in Proc. 3rd Scand. Workshop Algorithm Theory, Lecture Notes in Comput. Sci. 621, Springer-Verlag, Berlin, 1992, pp. 272–283.

[2] P. Bertolazzi, R. F. Cohen, G. Di Battista, R. Tamassia, and I. G. Tollis, *How to draw a series-parallel digraph*, Internat. J. Comput. Geom. Appl., (4) 1994, pp. 385–402.

[3] P. Bertolazzi and G. Di Battista, *On upward drawing testing of triconnected digraphs*, in Proc. 7th Annual ACM Sympos. Comput. Geom., 1991, ACM, New York, pp. 272–280.

[4] P. Bertolazzi, G. Di Battista, G. Liotta, and C. Mannino, *Upward drawings of triconnected digraphs*, Algorithmica, 12 (1994), pp. 476–497.

[5] R. Cole and U. Vishkin, *Approximate and exact parallel scheduling with applications to list, tree, and graph problems*, in Proc. 27th IEEE Symp. on Foundations of Computer Science, IEEE Computer Society Press, Los Alamitos, CA, 1986, pp. 478–491.

[6] P. Crescenzi, G. Di Battista, and A. Piperno, *A note on optimal area algorithms for upward drawings of binary trees*, Comput. Geom. Theory Appl., 2 (1992), pp. 187–200.

[7] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis, *Algorithms for drawing graphs: an annotated bibliography*, Comput. Geom. Theory Appl., 4 (1994), pp. 235–282.

[8] G. Di Battista, W. P. Liu, and I. Rival, *Bipartite graphs upward drawings and planarity*, Inform. Process. Lett., 36 (1990), pp. 317–322.

[9] G. Di Battista and R. Tamassia, *Algorithms for plane representations of acyclic digraphs*, Theoret. Comput. Sci., 61 (1988), pp. 175–198.

[10] G. Di Battista and R. Tamassia, *On-line graph algorithms with spqr-trees*, in Automata, Languages and Programming (Proc. 17th ICALP), Lecture Notes in Comput. Sci. 442, Springer-Verlag, New York, 1990, pp. 598–611.

[11] G. Di Battista and R. Tamassia, *On-line maintenance of triconnected components with SPQR-trees*, Algorithmica, 15 (1996), pp. 302–318.

[12] G. Di Battista, R. Tamassia, and I. G. Tollis, *Area requirement and symmetry display of planar upward drawings*, Discrete Comput. Geom., 7 (1992), pp. 381–401.

[13] P. Eades, T. Lin, and X. Lin, *Minimum size h-v drawings*, in Advanced Visual Interfaces (Proc. AVI '92), World Scientific Series in Computer Science 36, World Scientific, River Edge, NJ, 1992, pp. 386–394.

[14] D. Fussell, V. Ramachandran, and R. Thurimella, *Finding triconnected components by local replacements*, in Automata, Languages and Programming (Proc. 16th ICALP), Lecture Notes in Comput. Sci. 372, Springer-Verlag, New York, 1989, pp. 379–393.

[15] A. Garg, M. T. Goodrich, and R. Tamassia, *Area-efficient upward tree drawings*, in Proc. 9th Annual ACM Sympos. Comput. Geom., ACM, New York, 1993, pp. 359–368.

[16] A. Garg and R. Tamassia, *On the computational complexity of upward and rectilinear planarity testing*, in Proc. Graph Drawing '94, Lecture Notes in Comput. Sci. 894, Springer-Verlag, New York, 1995, pp. 286–297.

[17] J. Hopcroft and R. E. Tarjan, *Dividing a graph into triconnected components*, SIAM J. Comput., 2 (1973), pp. 135–158.

[18] J. Hopcroft and R. E. Tarjan, *Efficient planarity testing*, J. ACM, 21 (1974), pp. 549–568.

[19] M. D. Hutton and A. Lubiw, *Upward planar drawing of single source acyclic digraphs*, in Proc. 2nd ACM–SIAM Sympos. Discrete Algorithms, San Francisco, CA, SIAM, Philadelphia, 1991, pp. 203–211.

[20] M.-Y. Kao and G. E. Shannon, *Local reorientations, global order, and planar topology*, in Proc. 30th IEEE Symp. on Foundations of Computer Science, IEEE Computer Society Press, Los Alamitos, CA, 1989, pp. 286–296.

[21] D. Kelly, *Fundamentals of planar ordered sets*, Discrete Math., 63 (1987), pp. 197–216.

[22] D. Kelly and I. Rival, *Planar lattices*, Canad. J. Math., 27 (1975), pp. 636–665.

[23] A. Lempel, S. Even, and I. Cederbaum, *An algorithm for planarity testing of graphs*, in Theory of Graphs: Internat. Symposium, Rome, 1966, Gordon and Breach, New York, 1967, pp. 215–232.

[24] L. Lovasz and M. D. Pummer, *Matching Theory*, Ann. Discrete Math. 29, North–Holland, Amsterdam, 1986.

[25] A. Papakostas, *Upward planarity testing of outerplanar dags*, in Proc. Graph Drawing '94, Lecture Notes in Comput. Sci. 894, Springer-Verlag, New York, 1995, pp. 298–306.

[26] C. Platt, *Planar lattices and planar graphs*, J. Combin. Theory Ser. B, 21 (1976), pp. 30–39.

[27] V. Ramachandran and J. Reif, *Planarity Testing in Parallel*, Tech. Report TR-90-15, Department of Computer Science, University of Texas at Austin, 1990.

[28] V. Ramachandran and J. H. Reif, *An optimal parallel algorithm for graph planarity*, in

Proc. 30th Annual IEEE Sympos. Found. Comput. Sci., IEEE Computer Society Press, Los Alamitos, CA, 1989, pp. 282–293.

[29] E. Reingold and J. Tilford, *Tidier drawing of trees*, IEEE Trans. Software Eng., SE-7(1981), pp. 223–228.

[30] I. Rival, *Reading, drawing, and order*, in Algebras and Orders, I. G. Rosenberg and G. Sabidussi, eds., Kluwer Academic Publishers, Norwell, MA, 1993, pp. 359–404.

[31] K. J. Supowit and E. M. Reingold, *The complexity of drawing trees nicely*, Acta Inform., 18 (1983), pp. 377–392.

[32] R. Tamassia and J. S. Vitter, *Parallel transitive closure and point location in planar structures*, SIAM J. Comput., 20 (1991), pp. 708–725.

[33] C. Thomassen, *Planar acyclic oriented graphs*, Order, 5 (1989), pp. 349–361.

[34] W. T. Trotter and J. Moore, *The dimension of planar posets*, J. Combin. Theory Ser. B., 22 (1977), pp. 54–67.