ELSEVIER

Computational Geometry 13 (1999) 141–148

Computational Geometry
Theory and Applications

www. elsevier.nl/locate/comgeo

Further results on arithmetic filters for geometric predicates ☆

Olivier Devillers [a,*,1], Franco P. Preparata [b,2]

[a] *INRIA, BP 93, 06902 Sophia Antipolis, France*

[b] *Brown University, Department of Computer Science, Providence, RI 02912-1910, USA*

Communicated by K. Mehlhorn; submitted 9 December 1998; accepted 5 March 1999

Abstract

An efficient technique to solve precision problems consists in using exact computations. For geometric predicates, using systematically expensive exact computations can be avoided by the use of filters. The predicate is first evaluated using rounding computations, and an error estimation gives a certificate of the validity of the result. In this note, we study the statistical efficiency of filters for cosphericity predicate with an assumption of regular distribution of the points. We prove that the expected value of the polynomial corresponding to the insphere test is greater than ε with probability $O(\varepsilon \log 1/\varepsilon)$ improving the results of a previous paper. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Computational geometry; Delaunay triangulation; Exact arithmetic

---

1. Introduction

The assumption of real-number arithmetic, which is at the basis of conventional geometric algorithms, has been seriously challenged in recent years, since digital computers do not exhibit such capability. Geometric algorithms involve the evaluation of predicates; to guarantee the structural correctness of the results, predicates must be evaluated *exactly*. A geometric predicate usually consists of evaluating the sign of some algebraic expression. In most cases, rounded computations yield a reliable result, but sometimes rounded arithmetic introduces errors which may invalidate the algorithms. Assuming error-free input data, the rounded arithmetic may produce an incorrect result only if the exact absolute value of the algebraic expression is smaller than some (small) ε, which represents the largest error that may arise in the evaluation of the expression. The threshold ε depends on the structure of the expression and on the adopted computer arithmetic. This is basically the philosophy behind the notion of *arithmetic filters*,

---

☆ This work was partially supported by ESPRIT LTR GALIA and by the U.S. Army Research Office under grant DAAH04-96-1-0013.

* Corresponding author. E-mail: olivier.devillers@sophia.inria.fr

[1] This work was done in part while O. Devillers was visiting Brown University.

[2] E-mail: franco@cs.brown.edu

PII: S0925-7721(99)00011-5

whose function is to adjust the arithmetic overhead, so that no more effort is expended than required by the test instance.

It is therefore of interest to estimate the frequency with which recourse to arithmetic engines more powerful than standard platforms is necessary. Such analysis must be carried out by making some a priori hypothesis on the distribution of the input data, which are treated like random variables. Since for our objectives only the absolute value of the algebraic expressions is significant, hereafter “value” is to be intended as “absolute value”.

In a previous paper [1], we have carried out such analysis for two crucial geometric predicates, the orientation test (which-side of a hyperplane) and the insphere test (inside/ouside a hypersphere), on the hypotheses that the input points were uniformly distributed either in the unit ball $\mathcal{B}_\delta$ or in the unit cube $\mathcal{C}_\delta = [-1,1]^\delta$ in δ-dimensional space. Our results were that, for a small value V, the probability that the result of the orientation test is $< V$ is $\Theta(V)$ in all dimensions, whereas for the more complex insphere test we obtained bounds sublinear in V. Specifically, we obtained $\mathrm{O}(V^{2/3})$ in dimension 1 (which is tight), $\mathrm{O}(V^{1/2})$ in dimension 2, and $\mathrm{O}(V^{1/2}\ln V)$ in higher dimension.

Later on, we discovered a discrepancy between these theoretical findings for $\delta > 1$ and the results of extensive simulations, which seemed to exhibit a linear behavior (see below). This observation motivated a finer analysis, reported in this note, whose conclusion is that for $\delta > 1$ and for $\delta + 2$ points $p_1, p_2, \ldots, p_{\delta+2}$ uniformly chosen in the unit ball, the probability that the value of the determinant, embodying the insphere test of $p_{\delta+2}$ versus $p_1, p_2, \ldots, p_{\delta+1}$, is $< V$ is $\mathrm{O}(V\ln(1/V))$, in closer agreement with the simulations. The results extend to points uniformly chosen in a cube. We also present an application of this analysis to the three-dimensional insphere test carried out with floating point arithmetic.

2. Analysis of the insphere test

The algebraic expression embodying the predicate which tests if a point $p_{\delta+1}$ belongs to the sphere S passing through points $p_1, p_2, \ldots, p_\delta$ and the origin, is the following determinant [1]:

$$\Delta_\delta = \begin{vmatrix} x_{11} & x_{12} & \ldots & x_{11}^2 + x_{12}^2 + \cdots + x_{1\delta}^2 \\ x_{21} & x_{22} & \ldots & x_{21}^2 + x_{22}^2 + \cdots + x_{2\delta}^2 \\ \vdots & \vdots & \vdots & \vdots \\ x_{\delta+1,1} & x_{\delta+1,2} & \ldots & x_{\delta+1,1}^2 + x_{\delta+1,2}^2 + \cdots + x_{\delta+1,\delta}^2 \end{vmatrix}.$$

As mentioned in Section 1, in dimension 1 the insphere test reduces to an in-interval test and is only of moderate interest. Nevertheless, we have obtained the following tight bound [1]:

$$\mathrm{Prob}\big(|\Delta_1| \leqslant V\big) \leqslant \frac{17\sqrt[3]{2}}{4} V^{2/3} \simeq 5.36 V^{2/3}.$$

We now turn our attention to higher dimension, and let $c = (c_1/2, \ldots, c_\delta/2)$ denote the center of the sphere S. In the above determinant, subtracting column i times c_i from the last column, enables us to rewrite Δ_δ as

$$\Delta_\delta = \begin{vmatrix} x_{11} & x_{12} & \dots & x_{1\delta} & 0 \\ x_{21} & x_{22} & \dots & x_{2\delta} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{\delta,1} & x_{\delta,2} & \dots & x_{\delta,\delta} & 0 \\ x_{\delta+1,1} & x_{\delta+1,2} & \dots & x_{\delta+1,\delta} & W \end{vmatrix} \tag{1}$$

$$= |p_1 p_2 \dots p_\delta| W, \tag{2}$$

where

$$W = \left(x_{\delta+1,1}^2 + \dots + x_{\delta+1,\delta}^2\right) - \sum_{i=1}^{\delta} c_i x_{\delta+1,i}.$$

Adding and subtracting $\sum(c_i^2/4)$ from the last expression we obtain

$$W = \sum_{i=1}^{\delta} \left(x_{\delta+1,i} - \frac{c_i}{2}\right)^2 - \sum_{i=1}^{\delta} \left(\frac{c_i}{2}\right)^2.$$

This expression can be more synthetically rewritten as $W = |cp_{\delta+1}|^2 - r^2$, i.e., W is power$(p_{\delta+1}, S)$ of point $p_{\delta+1}$ with respect to the sphere S. Notice that power$(p_{\delta+1}, S)$ is positive if $p_{\delta+1}$ is external to S and negative if it is internal. Therefore, random variable Δ_δ is the product of the two random variables $|p_1 p_2 \dots p_\delta|$ and power$(p_{\delta+1}, S)$ (of which, incidentally, $|p_1 p_2 \dots p_\delta|$ has the form of a standard orientation test in dimension δ). Therefore, to complete our analysis we must

(1) analyze the statistical behavior of $|p_1 p_2 \dots p_\delta|$,
(2) analyze the statistical behavior of power$(p_{\delta+1}, S)$,
(3) obtain a convenient upper bound to the product of two random variables.

These tasks are the object of the next three subsections. The main idea of the proof is to use the fact that $W =$ power$(p_{\delta+1}, S)$ does not depend actually on $p_1, p_2, \dots, p_\delta$ but only on their circumscribing sphere.

2.1. Orientation test

In [1] we have shown that, given δ points uniformly distributed in the unit ball $\mathcal{B}_\delta$ in dimension δ,

$$\mathrm{Prob}\big(|p_1 p_2 \dots p_\delta| \leqslant V\big) \leqslant \sigma_\delta V,$$

where $\sigma_\delta = \delta v_{\delta-1}^{\delta} / v_\delta^{\delta-1}$ and v_j denotes the volume of the unit ball in dimension j.

In fact, these results can be extended without any difficulty to the case in which the value of $|p_1 p_2 \dots p_\delta|$ is constrained to an interval $[V, V + \mathrm{d}V]$, by simply changing in Eq. (c) of [1] the integration bounds from $\int_{a_\delta=0}^{\min(V,a_{\delta-1})}$ to $\int_{a_\delta=\min(V,a_{\delta-1})}^{\min(V+\mathrm{d}V,a_{\delta-1})}$. This trivial modification readily yields

$$\mathrm{Prob}\big(V \leqslant |p_1 p_2 \dots p_\delta| \leqslant V + \mathrm{d}V \mid p_1, p_2, \dots, p_\delta \in \mathcal{B}_\delta\big) \leqslant \sigma_\delta \, \mathrm{d}V. \tag{3}$$

This result generalizes to the uniform distribution in the unit cube $\mathcal{C}_\delta = [-1, 1]^\delta$ as in [1].

$$\mathrm{Prob}\big(V \leqslant |p_1 p_2 \dots p_\delta| \leqslant V + \mathrm{d}V \mid p_1, p_2, \dots, p_\delta \in \mathcal{C}_\delta\big) \leqslant \psi_\delta \, \mathrm{d}V, \tag{4}$$

where

$$\psi_\delta = \frac{\delta v_\delta v_{\delta-1}^{\delta} \delta^{\delta(\delta-1)/2}}{2^{\delta^2}}.$$

2.2. Power of a point with respect to a sphere

Given a sphere S, with center c and radius r, we wish to compute the probability for a random point p to have a small (absolute value) power with respect to S.

For a small value V we observe that

$$\left(\left|\mathrm{power}(p,S)\right| = \left||cp|^2 - r^2\right| \leqslant V\right) \Longrightarrow \left(r - \frac{V}{2r} \simeq \sqrt{r^2 - V} \leqslant |cp| \leqslant \sqrt{r^2 + V} \leqslant r + \frac{V}{2r}\right).$$

Therefore, the value of the power of p with respect to S is smaller than V if p belongs to a spherical crown of S of width V/r. Clearly, the volume of such crown is equivalent to the measure (area) of S multiplied by V/r, i.e., it is given by $\delta v_\delta r^{\delta-1} V/r = \delta v_\delta r^{\delta-2} V$ (this holds in our hypothesis of small V).

Thus Prob(power$(p,S) \leqslant V$) is bounded as follows:

$$\mathrm{Prob}\left(\mathrm{power}(p,S) \leqslant V\right) \leqslant \frac{\mathrm{volume}(\mathrm{crown} \cap \Omega)}{\mathrm{volume}(\Omega)}.$$

The term volume(crown $\cap\, \Omega$) is the product of V/r by the area of $S \cap \Omega$. At this point we assume $\Omega \subset \mathcal{C}_\delta$, which is obviously verified when Ω is either $\mathcal{B}_\delta$ or $\mathcal{C}_\delta$. If $r < 1$ we bound from above the volume of the crown by $\delta v_\delta r^{\delta-2} V \leqslant \delta v_\delta V$. If $r \geqslant 1$ we restrict ourselves to the portion of the crown internal to $\mathcal{C}_\delta$ and obtain $\mathrm{area}(S \cap \mathcal{C}_\delta) V/r \leqslant \mathrm{area}(S \cap \mathcal{C}_\delta) V \leqslant \delta v_\delta V$.

In conclusion, we have

$$\mathrm{Prob}\left(\mathrm{power}(p,S) \leqslant V \mid S \text{ given};\ p \in \mathcal{B}_\delta\right) \leqslant \frac{\delta v_\delta}{v_\delta} V = \delta V, \tag{5}$$

$$\mathrm{Prob}\left(\mathrm{power}(p,S) \leqslant V \mid S \text{ given};\ p \in \mathcal{C}_\delta\right) \leqslant \frac{\delta v_\delta}{2^\delta} V. \tag{6}$$

2.3. Product of two random variables

To complete the analysis outlined above, we need a technical result concerning the probability of a product of random variables.

Let a and b be two random variables such that the marginal probability of a satisfies Prob($V \leqslant a \leqslant V + \mathrm{d}V$) $\leqslant A\,\mathrm{d}V$ and the probability of b conditional on a satisfies Prob($b \leqslant V \mid a$) $\leqslant BV$, for some constants A and B. Notice that our random variables $|p_1 p_2 \ldots p_\delta|$ and power(p,S) fit the specifications of a and b, respectively. We shall bound from above the event $ab < V$ by a union of events of the kind $\alpha \leqslant a \leqslant \alpha + \mathrm{d}\alpha$ and $b \leqslant V/\alpha$, as illustrated in Fig. 1.

Thus we have

$$\begin{aligned}
\mathrm{Prob}(ab \leqslant V) &\leqslant \mathrm{Prob}(a \leqslant V) + \int_V^1 \mathrm{Prob}(a = \alpha)\mathrm{Prob}\left(b \leqslant \frac{V}{\alpha} \mid a = \alpha\right)\mathrm{d}\alpha + \mathrm{Prob}(b \leqslant V) \\
&\leqslant (A + B)V + \int_V^1 ABV \frac{\mathrm{d}\alpha}{\alpha} \\
&\leqslant (A + B)V + ABV \ln\frac{1}{V}.
\end{aligned} \tag{7}$$

Notice that for A and B both $\geqslant 2$ and for $V \leqslant 1/\mathrm{e}$, the first term is dominated by the second one.

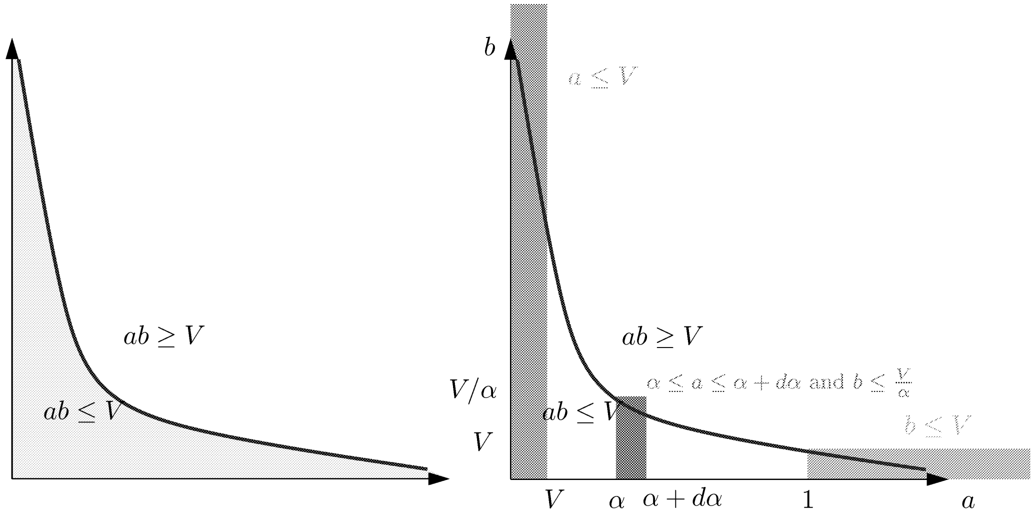

Fig. 1. Upper bounding event $ab \leqslant V$.

3. Completing the analysis

In this section, we present the main conclusion of this note. Recalling that

$$\Delta_\delta = |p_1 p_2 \ldots p_\delta| \cdot \mathrm{power}\bigl(p_{\delta+1}, \mathrm{sphere}(p_1 p_2 \ldots p_\delta)\bigr),$$

and the previous bounds, we obtain for the two domains:

$$\mathrm{Prob}\bigl(\Delta_\delta \leqslant V \mid p_1, \ldots, p_{\delta+1} \in \mathcal{B}_\delta\bigr) \leqslant (\sigma_\delta + \delta)V + \sigma_\delta \delta V \ln \frac{1}{V}, \tag{8}$$

$$\mathrm{Prob}\bigl(\Delta_\delta \leqslant V \mid p_1, \ldots, p_{\delta+1} \in \mathcal{C}_\delta\bigr) \leqslant \frac{\delta v_\delta \psi_\delta}{2^\delta} V \ln \frac{1}{V} + \left(\psi_\delta + \frac{\delta v_\delta}{2^\delta}\right) V, \tag{9}$$

which express a bound nearly linear in V for the absolute value of the incircle test for $\delta > 1$.

For small values of δ we recall from [1] the (approximate) values of v_δ, σ_δ and ψ_δ:

δ	v_δ	σ_δ	ψ_δ
1	2	1	1
2	π	$\frac{8}{\pi} \simeq 2.5$	$\pi \simeq 3.1$
3	$\frac{4\pi}{3} \simeq 4.2$	$\simeq 5.3$	$\simeq 21$
4	$\frac{\pi^2}{2} \simeq 4.9$	$\simeq 10$	$\simeq 380$
5	$\frac{8\pi^2}{15} \simeq 5.3$	$\simeq 19$	$\simeq 22.000$
6	$\frac{\pi^3}{6} \simeq 5.2$	$\simeq 35$	$\simeq 4.500.000$

$$\begin{aligned}
&\mathrm{Prob}\big(\Delta_2 \leqslant V \mid p_1,\ldots,p_3 \in \mathcal{B}_2\big) \leqslant 5.0V \ln\frac{1}{V} + 4.5V,\\
&\mathrm{Prob}\big(\Delta_3 \leqslant V \mid p_1,\ldots,p_4 \in \mathcal{B}_3\big) \leqslant 16V \ln\frac{1}{V} + 8V,\\
&\mathrm{Prob}\big(\Delta_4 \leqslant V \mid p_1,\ldots,p_5 \in \mathcal{B}_4\big) \leqslant 40V \ln\frac{1}{V} + 14V,\\
&\mathrm{Prob}\big(\Delta_5 \leqslant V \mid p_1,\ldots,p_6 \in \mathcal{B}_5\big) \leqslant 95V \ln\frac{1}{V} + 24V,\\
&\mathrm{Prob}\big(\Delta_6 \leqslant V \mid p_1,\ldots,p_7 \in \mathcal{B}_6\big) \leqslant 207V \ln\frac{1}{V} + 40V
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Prob}\big(\Delta_2 \leqslant V \mid p_1,\ldots,p_3 \in \mathcal{C}_2\big) \leqslant 4.9V \ln\frac{1}{V} + 4.7V,\\
&\mathrm{Prob}\big(\Delta_3 \leqslant V \mid p_1,\ldots,p_4 \in \mathcal{C}_3\big) \leqslant 32V \ln\frac{1}{V} + 22V,\\
&\mathrm{Prob}\big(\Delta_4 \leqslant V \mid p_1,\ldots,p_5 \in \mathcal{C}_4\big) \leqslant 468V \ln\frac{1}{V} + 381V,\\
&\mathrm{Prob}\big(\Delta_5 \leqslant V \mid p_1,\ldots,p_6 \in \mathcal{C}_5\big) \leqslant 18000V \ln\frac{1}{V} + 22000V,\\
&\mathrm{Prob}\big(\Delta_6 \leqslant V \mid p_1,\ldots,p_7 \in \mathcal{C}_6\big) \leqslant 2.200.000V \ln\frac{1}{V} + 4.500.000V.
\end{aligned}$$

These analytical results can be compared with the experimental results mentioned earlier. The latter have been obtained using random point selection in $\mathcal{B}_\delta$, and are shown in Fig. 2. They confirm the sublinear behavior for $\delta = 1$ and a basically linear behavior for $\delta \geqslant 2$ near $V = 0$. However, the constants

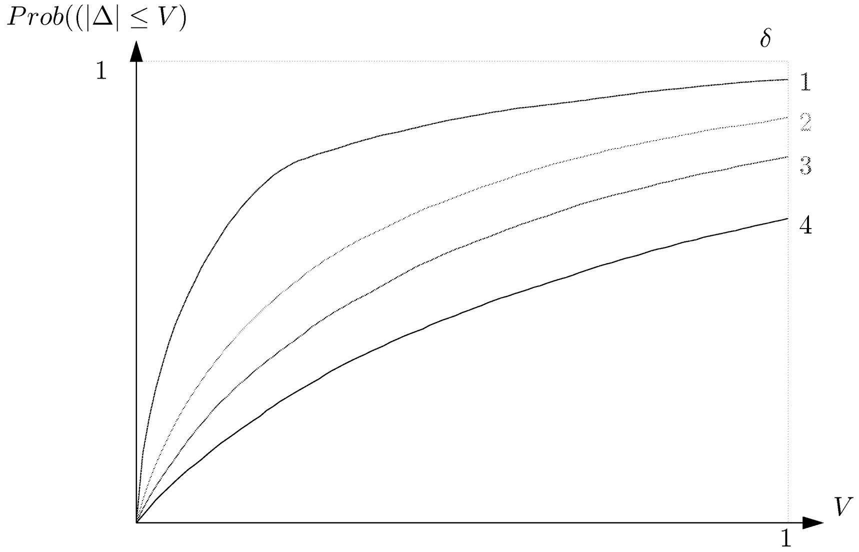

Fig. 2. Experimental results on random incircle tests.

reported above are far from tight when the dimension increases, which is a clear byproduct of the technique of proof used in [1] to bound ψ_δ.

4. Example: 3D insphere test with double precision floating point arithmetic

We now consider a practical implementation of the insphere test in three dimensions. The corresponding expression is given below. We assume that entries (point coordinates) are floating point numbers in the range $[-1, 1]$ and that they are stored as double precision numbers with a 53-bit mantissa. We assume that the computation complies with the IEEE 754 norm.

We first detail the formula for the insphere test:

$$\begin{aligned}\begin{vmatrix} x_1 & y_1 & z_1 & x_1^2+y_1^2+z_1^2 \\ x_2 & y_2 & z_2 & x_2^2+y_2^2+z_2^2 \\ x_3 & y_3 & z_3 & x_3^2+y_3^2+z_3^2 \\ x_4 & y_4 & z_4 & x_4^2+y_4^2+z_4^2 \end{vmatrix} &= -\left(x_1^2+y_1^2+z_1^2\right)\begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} + \left(x_2^2+y_2^2+z_2^2\right)\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \\ &\quad - \left(x_3^2+y_3^2+z_3^2\right)\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix} + \left(x_4^2+y_4^2+z_4^2\right)\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix},\end{aligned}$$

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = x_1(y_2z_3 - y_3z_2) - x_2(y_1z_3 - y_3z_1) + x_3(y_1z_2 - y_2z_1).$$

We now estimate the maximum a priori round-off error using the following standard rules: $\text{error}(x + y) \leqslant \text{error}(x) + \text{error}(y) + (x + y)2^{-54}$ and $\text{error}(xy) \leqslant x \cdot \text{error}(y) + y \cdot \text{error}(x) + xy2^{-54}$. Each computation is analyzed in terms of the elementary operations of addition/subtraction or multiplication.

Ref.	Description	Typical expression	Upper bound	Error bound
$\boxed{1}$	entry	x_1	1	2^{-54}
$\boxed{2}$	$\boxed{1} \times \boxed{1}$	y_2z_3	1	$3 \cdot 2^{-54}$
$\boxed{3}$	$\boxed{2} + \boxed{2}$	$y_2z_3 - y_3z_2$	2	$2 \cdot 3 \cdot 2^{-54} + 2 \cdot 2^{-54} = 2^{-51}$
$\boxed{4}$	$\boxed{1} \times \boxed{3}$	$x_1(y_2z_3 - y_3z_2)$	2	$2^{-51} + 2 \cdot 2^{-54} = 5 \cdot 2^{-53}$
$\boxed{5}$	$\boxed{4} + \boxed{4}$		4	$2 \cdot 5 \cdot 2^{-53} + 4 \cdot 2^{-54} = 3 \cdot 2^{-51}$
$\boxed{6}$	$\boxed{5} + \boxed{4}$	$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$	6	$3 \cdot 2^{-51} + 5 \cdot 2^{-53} + 6 \cdot 2^{-54} = 5 \cdot 2^{-51}$
$\boxed{7}$	$\boxed{2} + \boxed{3}$	$x_1^2 + y_1^2 + z_1^2$	3	$3 \cdot 2^{-54} + 2^{-51} + 3 \cdot 2^{-54} = 7 \cdot 2^{-53}$
$\boxed{8}$	$\boxed{6} \times \boxed{7}$		18	$6 \cdot 7 \cdot 2^{-53} + 3 \cdot 5 \cdot 2^{-51} + 18 \cdot 2^{-54} = 111 \cdot 2^{-53}$
$\boxed{9}$	$\boxed{8} + \boxed{8}$		36	$2 \cdot 111 \cdot 2^{-53} + 36 \cdot 2^{-54} = 120 \cdot 2^{-52}$
$\boxed{10}$	$\boxed{9} + \boxed{9}$	incircle test	72	$2 \cdot 120 \cdot 2^{-52} + 72 \cdot 2^{-54} = 129 \cdot 2^{-51} \simeq 2^{-44}$

If the points are uniformly distributed in the unit cube and snap-rounded to the nearest representable point, then the above calculations show that if the insphere test gives a result larger than $129 \cdot 2^{-51}$ (in absolute value), then its sign is reliable.

For simple precision numbers with 24 bits of mantissa, an analogous statement can be made for results larger than $129 \cdot 2^{-22} \simeq 2^{-15}$.

These results enable us to estimate the probability of failure of such filter, i.e.,

$$\text{prob}(\text{failure}) \leqslant 32(V) \ln \frac{1}{V} + 22(V),$$

with $V = 129 \cdot 2^{-51}$ or $V = 129 \cdot 2^{-22}$ for the two cases.

Claim. *If the absolute value of the insphere test in three dimensions for points in the unit cube computed with* 53 (*respectively* 24) *bit arithmetic is larger than* $129 \cdot 2^{-51} \leqslant 6 \cdot 10^{-14}$ (*respectively* $129 \cdot 2^{-22} \simeq 3 \cdot 10^{-5}$) *then the sign is reliable. The probability of failure of the certifier is less than* $6 \cdot 10^{-11}$ (*respectively* 0.011).

References

[1] O. Devillers, F. Preparata, A probabilistic analysis of the power of arithmetic filters, Discrete Comput. Geom. 20 (1998) 523–547. http://www.inria.fr/prisme/publis/dp-papaf-98.ps.gz