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Accurate cylindricity evaluation with axis-estimation preprocessing

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Abstract

Evaluating cylindricity is a very important application in metrology. In this paper, we focus on cylindricity evaluation based on radial form measurements. The standard characterization of cylindricity is the notion of *zone cylinder*, i.e. the cylindrical crown contained between two coaxial cylinders with minimum radial separation and containing all the data points. Unfortunately, the construction of the zone cylinder is a very complex geometric problem, which can be formulated as a nonlinear optimization. Recently a new method (referred to here as the hyperboloid method) has been discussed, which avoids the direct construction of the zone cylinder of a point set, but approximates it with guaranteed accuracy through a computationally very efficient iterative process based on a linearization of the underlying problem. The iterations can be viewed as the construction of a sequence of "zone hyperboloids" tending to the desired "zone cylinder." An important requirement of the method, however, is that the initial position of the cylindrical specimen axis be nearly vertical, since significant deviations from this condition essentially invalidate the process. It is the purpose of this paper to remove this shortcoming of the hyperboloid technique by providing a simple procedure for appropriately initializing the data (axis estimation). Axis estimation and the hyperboloid technique constitute an integrated methodology for cylindricity evaluation, which is currently the most effective. The theoretical foundations of the method are reviewed from a viewpoint that highlights its essential features and intuitively explains its effectiveness. The analytical discussion is complemented by experimental data concerning a few significant samples.

Keywords: Computational metrology; Cylindricity; Edge straightness; Hyperboloid; Method; Axis alignment

1. Introduction

Evaluating cylindricity is a very important application in metrology, since cylindrical surfaces are ubiquitous in industrial machining, and the realization of high-quality cylinders is a crucial technological objective [4]. This leads to stringent tolerancing and, consequently, a high premium is placed on accurate evaluations of cylindricity to avoid costly rejections of valid specimens.

Various measures of cylindricity have been proposed over the years, such as the "minimum-enclosing/enclosed cylinders," "least-square cylinder," etc. [1–3,5,6]. In addition, various data acquisition (measurement) protocols have been suggested, concerning the geometric configuration of the data points: on radial sections, on generatrices, on a bird-cage (a combination of radial sections and generatrices).

The consensus, however, appears to be that the characterization of cylindricity is the notion of *zone cylinder*, i.e. the cylindrical crown contained between two coaxial cylinders with minimum radial separation and containing all the data points. Moreover, it is desirable that no constraint be applied to the configuration of data points: a method should be capable of accepting any configuration of data points (although data acquisition instruments are likely to produce a regular sampling of the specimen surface).

Unfortunately, the construction of the zone cylinder is a very complex geometric problem, which can be formulated as a nonlinear optimization. It can be shown that constructing the zone cylinder through six points in space involves the solution of a system of six degree-4 equations, a task requiring sophisticated algebraic geometry tools [7]. The inherent difficulty is the determination of the axis of the zone cylinder. In fact, in the hypothesis that the axis is known, the zone-cylinder problem is reduced to its two-dimensional instance (the zone-circle problem), which is relatively easy to solve.

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Due to the substantial geometric and algorithmic difficulty of the determination of the zone cylinder, it is a frequent metrological practice to resort to simpler, but necessarily less effective, surrogates, such as least-square fit or the zone-circle of a two-dimensional projection of the data points parallel to the nominal axis of the cylinder [5].

In the recent past, a method for the construction of the zone cylinder has been proposed as an optimization task [8,9], by linearizing the underlying nonlinear problem. Except for the observation that the approach outperformed previous methods (a fact documented in those papers by several experiments), no analysis of efficacy or convergence was presented. In addition, in order to initialize the procedure, an estimate of the axis was assumed to be independently available. More recently [10] a similar algorithm has been discussed in a geometric setting that fully reveals the structure of the problems, characterizes its convergence, and provides an analytical criterion for assessing the quality of the result. This novel interpretation presents the successive iterations of the algorithm as the construction of a sequence of "zone hyperboloids" tending to the desired "zone cylinder." The analysis illustrates that the axis of the zone hyperboloid, constructed at each iteration of the algorithm, tends to the unknown axis of the zone cylinder, in such a fashion that, when the two axes coincide, so do the zone hyperboloid and the zone cylinder. When this happens, the radial separation of the zone hyperboloid becomes the radial separation of the zone cylinder, i.e. the cylindricity. The method, extensively discussed in [10], is briefly summarized, and in some sense significantly reinterpreted in Appendix A of this paper for the sake of self-containment. It suffices to recall here the main ideas.

A cylinder of near-vertical axis is conceptually replaced by a one-sheet hyperboloid with circular horizontal sections, sharing the same axis. Correspondingly, a zone cylinder is replaced by a zone hyperboloid, analogously defined. This associated hyperboloid is a much simpler object to manipulate from a computational standpoint. Clearly, the cylinder and its associated hyperboloid are two distinct geometric objects, but, if the common axis is known and brought to coincide with the z-axis of the frame of reference, cylinder and hyperboloid are provably the same object.

This observation is the key to the approach, since, rather than the set of zone cylinders, one must search the much more manageable set of zone hyperboloids. As alluded to earlier, the latter search is computationally much simpler, since it is implemented by linear programming in six-dimensions. The computation runs in time nearly proportional to the size of the data point set, and determines the zone hyperboloid with minimum radial separation. As it normally happens, the axis of the minimum zone cylinder of the given point set (although "nominally" vertical) is not exactly vertical and so is the axis of the computed zone hyperboloid. Therefore, we take the axis of the computed zone hyperboloid as a first approximation to the sought axis and subject the point set to a rigid motion bringing the z-axis to coincide with the axis of the zone hyperboloid. By this coordinate transformation the (unknown) axis of the minimum-zone cylinder is brought closer to the z-axis than it was initially. Iterating this process, we can approach the cylinder axis with excellent precision. Typically, two to three iterations obtain a satisfactory solution of the original problem. For more details about the approximation of zone cylinders by zone hyperboloids, the reader is referred to Appendix A.

It is important to stress that the homing of the zone-hyperboloid axis to the zone-cylinder axis is the crucial feature of the technique, rather than the decrease of the hyperboloid radial separation as the iterations proceed. In fact, as evidenced by experiments reported in Section 4, the hyperboloid radial separation may occasionally increase in the iterative process. This behavior is due to the fact that associated with a zone hyperboloid (of radial separation δ) there is a coaxial zone cylinder (of radial separation δ^*, see Appendix A) satisfying the inequality

$$\delta^* \leq \delta + \frac{5\epsilon^2}{R}$$

where R is the nominal radius and ε is a suitably defined axis-misalignment error. Thus, the convergence criterion is the reduction of ε. Indeed, $\epsilon = 0$ implies $\delta^* = \delta$.

Finally, an important operational requirement of the method (as also noted in [8,9]), is that the initial position of the axis be known. Significant deviations from this condition essentially invalidate the process, since iterative convergence cannot be established. This difficulty was avoided in [8,9] by assuming an independent good estimate of the axis as a precondition. Although in practice one may assume that the experimental setting assures the near-verticality of the axis, such mixture of algorithmic and manual resources is both unreliable and technically objectionable. It is an objective of this paper to remove this shortcoming by providing a simple procedure for appropriately initializing the data. This preprocessing step, referred to as *axis-estimation*, and the hyperboloid method, jointly constitute an integrated methodology for cylindricity evaluation. While the hyperboloid method is reviewed in Appendix A, the next sections discuss and analyze the axis-estimation. The analytical discussion is complemented by experimental data concerning a few significant samples. We conclude with some methodological observations and with a comment on the relevance of the approach to straightness evaluation.

2. An estimate of the axis

The physical object is nominally a portion of a circular cylinder between two plane faces normal to the axis (the distance between the delimiting faces is called the specimen's height). It is represented as a collection of points obtained by sampling the cylindrical surface. In principle, the samples could be randomly distributed, under some criteria of uniformity and coverage. Indeed, ours is the least constraining data sampling protocol, allowing any sufficiently dense placement of the points on the surface. In metrological practice, it is reasonable to assume that the data points belong to sections nominally perpendicular to the axis, because of the nature of instrumentation, although no constraint is placed on the spacing of points in a section or the inter-section spacing.

The axis-estimation procedure should be insensitive to the form-factor of the artefact (ratio of radius to height). Direct processing of the measured points, however, does not meet such criterion, since the form-factor significantly affects the robustness of the estimate. To avoid this difficulty, we propose the following approach.

Let S be the set of sample points. We construct the convex hull CH(S) of S and obtain its set F of facets, typically a triangulation of the surface. For each $f \in F$, we consider its unit normal $\boldsymbol{n}(f)$ and the set $\mathcal{M} = \{\boldsymbol{n}(f) : f \in F\}$, known as the Gaussian map of CH(S) (whose vertices are a set of points on the unit sphere centered at the origin). Typically, in $\mathcal{M}$ we shall recognize three clusters of points, descriptively denoted as follows: two "polar bundles" corresponding to convex hull facets associated with the faces of the cylinder and an "equatorial wheel" corresponding to facets associated with the cylindrical surface (see Fig. 1 for a pictorial illustration of these notions). Of course, we shall expect the presence of other unclustered normals, generally corresponding to the transitions between the plane faces and the cylindrical surface.

Consider a line L by the origin parallel to the unit vector (l, m, n) with $l^2 + m^2 + n^2 = 1$. The square distance of a point (x, y, z) from L is given by $(lx + my + nz)^2$. Let $\mathcal{M} = \{(x_i, y_i, z_i) : i = 1, \ldots, N\}$. The unit vectors (l, m, n) which extremize the quantity

$$\sum_{i=1}^{N}(lx_i + my_i + nz_i)^2$$

define, as is well known, the principal axes of inertia of the unit-mass point set $\mathcal{M}$. One of these axes is an estimate of the cylinder axis and its determination is, therefore, our objective.

To obtain the extremizing (l, m, n) subject to the constraint $l^2 + m^2 + n^2 = 1$, we introduce the Lagrange multiplier λ and seek the extrema of the function

$$\phi = \sum_{i=1}^{N}(lx_i + my_i + nz_i)^2 + \lambda(l^2 + m^2 + n^2)$$

The ensuing conditions

$$\frac{\partial\phi}{\partial l} = 0, \quad \frac{\partial\phi}{\partial m} = 0, \quad \frac{\partial\phi}{\partial n} = 0$$

result in the linear equations

$$\begin{vmatrix} X+\lambda & U & V \\ U & Y+\lambda & W \\ V & W & Z+\lambda \end{vmatrix} \times \begin{vmatrix} l \\ m \\ n \end{vmatrix} = 0$$

where

$$X = \sum_{i=1}^{N} x_i^2, \quad Y = \sum_{i=1}^{N} y_i^2, \quad Z = \sum_{i=1}^{N} z_i^2$$

and

$$U = \sum_{i=1}^{N} x_i y_i, \quad V = \sum_{i=1}^{N} x_i z_i, \quad W = \sum_{i=1}^{N} y_i z_i$$

As expected, the sought vectors are the eigenvectors of the above matrix; in the language of theoretical mechanics, the eigenvalues of the matrix are the inverses of the square principal moments of inertia. By "axial eigenvalue" we denote the one whose associated eigenvector is directed like the cylinder axis.

The determination of the axial eigenvalue and of the corresponding eigenvector (the preliminary axis estimate) is a selection among three alternatives, which is carried out according to a very simple, and natural, criterion:

1. Four sample points p_1, p_2, p_3, p_4 are chosen at random from the data set.
2. For each eigenvector (l, m, n) (of the three alternative eigenvectors), points p_1, p_2, p_3, p_4 are projected onto a plane π, orthogonal to (l, m, n). These four projections are tested for co-circularity.

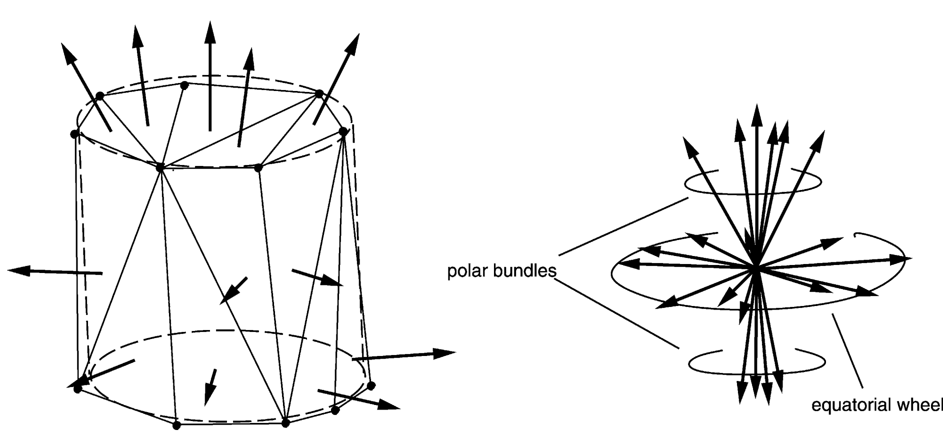

Fig. 1. The convex hull of a set of points on a nominally cylindrical surface and its Gaussian map. Notice polar bundles and equatorial wheel.

3. The selected eigenvector is the one which best satisfies the co-circularity test.

We now describe the co-circularity test. Again, let (l, m, n) be a chosen direction in space. We seek a transformation of coordinates from (x, y, z) to (X, Y, Z) which brings the Z-axis to coincide with direction (l, m, n) in the original reference system. By standard techniques of analytic geometry, we obtain the transformation (projection) expressed by the following matrix (where we have set $\rho^2 = m^2 + n^2$):

$$M = \begin{pmatrix} \rho & -\dfrac{lm}{\rho} & -\dfrac{nl}{\rho} \\ 0 & \dfrac{n}{\rho} & -\dfrac{m}{\rho} \\ l & m & n \end{pmatrix} \tag{1}$$

This transformation, which projects a point p_i to a plane π orthogonal to vector (l, m, n), is just one among all the projections which can be obtained from it by an additional arbitrary rotation around the Z-axis of reference system (X, Y, Z). So if $p_i = (x_i, y_i, z_i)$ is projected to (X_i, Y_i, Z_i), we have

$$X_i = \rho x_i - \frac{lm}{\rho} y_i - \frac{nl}{\rho} z_i$$
$$Y_i = \frac{n}{\rho} y_i - \frac{m}{\rho} z_i$$
$$Z_i = 0$$

Finally the four projections are co-circular if and only if

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ X_1 & X_2 & X_3 & X_4 \\ Y_1 & Y_2 & Y_3 & Y_4 \\ X_1^2 + Y_1^2 & X_2^2 + Y_2^2 & X_3^2 + Y_3^2 & X_4^2 + Y_4^2 \end{vmatrix} = 0 \tag{2}$$

Since, in general, data point are not exactly on a cylinder surface, none of the three co-circularity evaluations will be exactly equal to 0. We shall, therefore, choose the one with the smallest absolute value of the co-circularity determinant.

3. Implementation of the axis-estimation

The objective of our application is to make the axis estimate independent of the accidental physical orientation of the cylinder axis, due both to incorrect placement of the specimen on the measuring platform and to imperfect machining (non-orthogonality between end-faces and axis). From the preceding discussion, it is clear that identification of the polar bundles, and their removal from the Gaussian map, will make the axis estimate depend exclusively upon the equatorial wheel, as desired.

To this end, we have implemented the following procedure:

1. Place the specimen nominally vertical on the measurement platform.
2. Collect measurements along a set of sections nominally orthogonal to the axis of the platform (this is not a requirement and simply reflects common measuring practice).
3. Compute the convex hull of the obtained data points, and form its Gaussian map $\mathcal{M}$.
4. (Filter) Compute the principal axes of inertia of $\mathcal{M}$ and select the axial eigenvector $\boldsymbol{n}$.
5. Remove from $\mathcal{M}$ all normals whose inner product with $\boldsymbol{n}$ is larger than some threshold θ (typically $\theta = 0.15$; this is aimed at the removal of the polar bundles).
6. Repeat Step 4 on the reduced set of normals: the resulting axial eigenvector $\boldsymbol{n}^*$ is the desired estimate.

The main objective of this analysis is to elucidate the interplay of the various selectable parameters in order to make recommendations for a measurement procedure that would optimize axis identification. We have considered the following parameters, which characterize the metrological setting:

- N': number of sample points per measurement sections.
- s: number of sections.

In addition, we have considered parameters intended to describe the randomness of individual measurements, relating to the quality of the measured specimen and its physical placement in the measuring apparatus:

- t: initial tilt of the specimen on the platform, measured in degrees.
- p: perturbation parameter, intended to reflect the quality of the specimen. Each sample point, nominally generated on the cylindrical surface, is subjected to a random radial displacement, uniformly distributed with mean 0 and standard deviation p.

The quality of the axis estimation is expressed by the parameter ε^*, *estimation error*, which is the angle in radians between the true axis and the axis $\boldsymbol{n}^*$ estimated by our procedure.

Fig. 2 shows that the estimation error grows more or less linearly with the intensity of the perturbation. It also shows that this growth is independent of the initial tilt t, as fully expected, since theoretically axis determination is independent of the frame of reference. Thus, hereafter we shall refer to $t = 0$.

Next we explore how the sampling protocol affects the quality of the estimation. Specifically, let R and h be the nominal radius and height of the cylindrical specimen, respectively. We have the choice of conducting our sampling on a variable number of sections, with the only condition that each section contain roughly the same number of samples. Intuitively, a larger section spacing should improve the accuracy, since the “normals” are likely to be nearly orthogonal to the cylinder surface (the spacing between adjacent sections moderates the effect of the perturbation p); on the other hand, a larger number of sections increases the number of “normals,” thereby compensating through the sample size the fluctuations due to the perturbation. Indeed, the simulations support the outlined intuition. In Figs. 3 and 4, we display the final estimation error as a function of p for different numbers of sections for two specimens

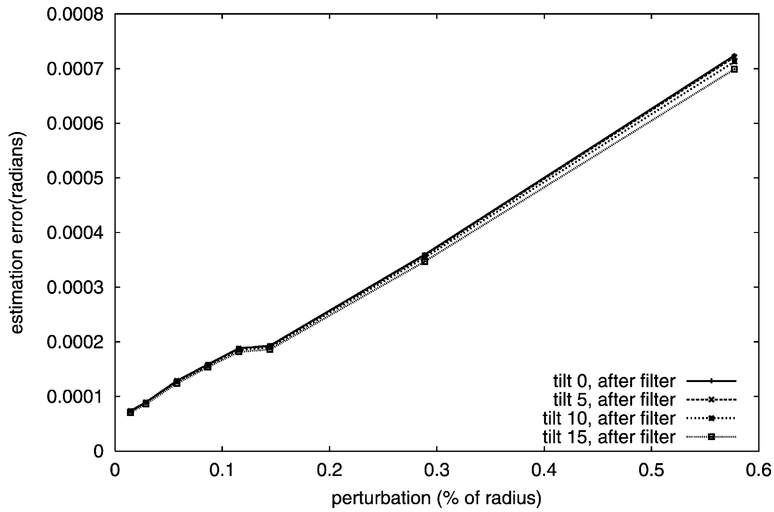

Fig. 2. Estimation error as a function of perturbation p for different values of the initial specimen tilt, with $N' = 500$, $s = 6$, and section spacing $2R$.

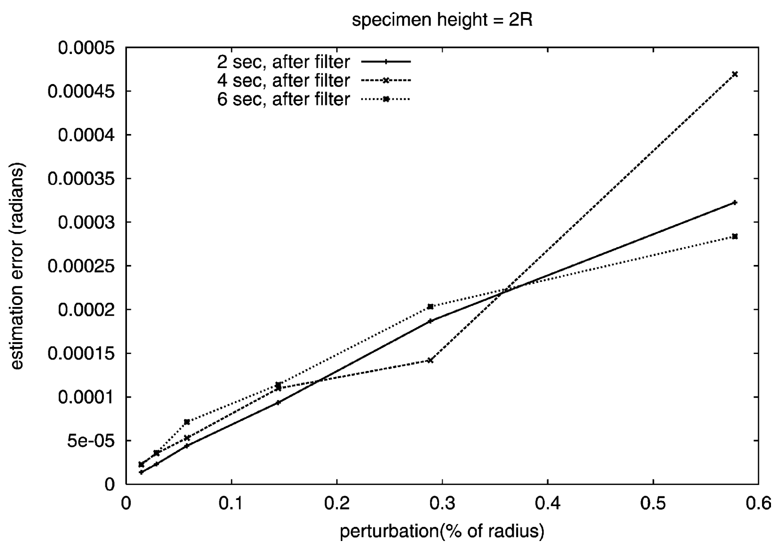

Fig. 3. Estimation error as a function of perturbation p for different values of the number of sections, with $t = 0$, $N' = 500$, and section spacing $2R$.

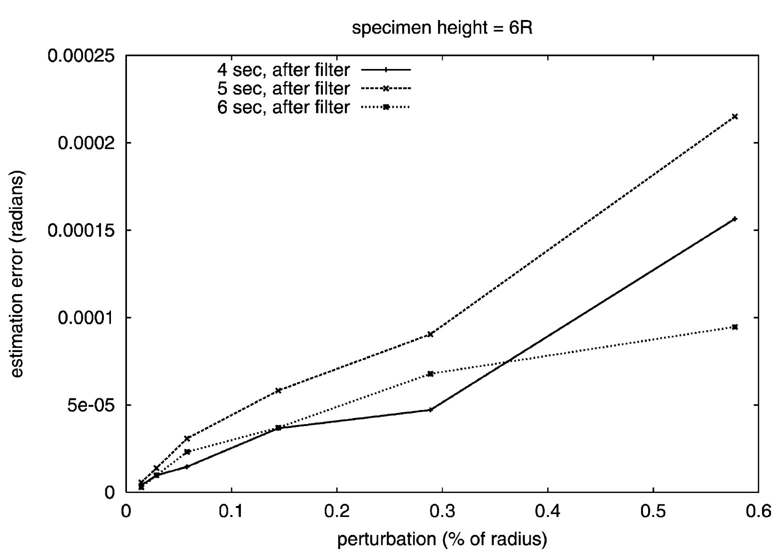

Fig. 4. Estimation error as a function of perturbation p for different values of the number of sections, with $t = 0$, $N' = 500$, and section spacing $6R$.

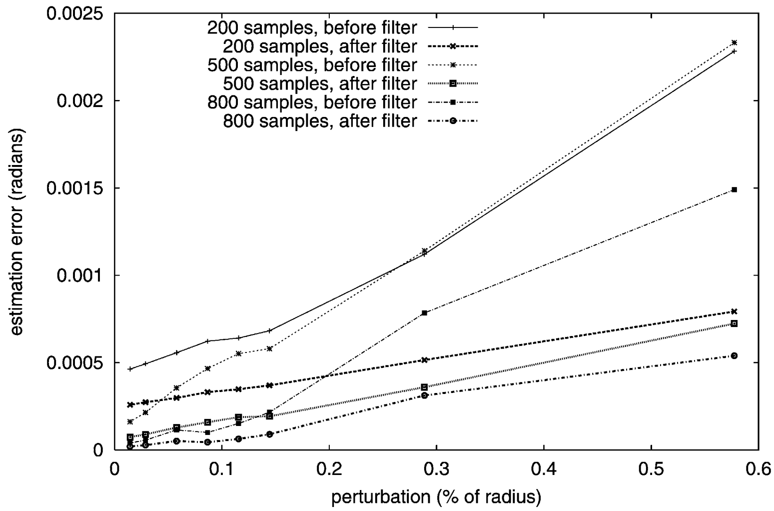

Fig. 5. Estimation error as a function of perturbation p for different values of the sample size, distributed on six sections, before and after the filtering operation, for $t = 0$, and section spacing $2R$.

Table 1

Specimen	Initial	Iteration 1	Iteration 2	Iteration 3	Radius	Cylindricity	LS cyl.
CF2	0.0702	5.52×10^{-5}	1.2×10^{-9}	0	59.98	0.18395	0.2119
CF3	25.05	3.82×10^{-3}	3.78×10^{-7}	0	49.99	0.009409	0.01037
brXYZ	8.823×10^{-4}	9.49×10^{-7}	4.32×10^{-13}	0	1.4995	0.002614	0.003642
dkXYZ		3.15×10^{-4}	2.09×10^{-9}	0	0.6243	0.0000157	0.0000248

of very different form-factors ($h/R = 2, 6$), and observe that the quality is basically independent of the number of sections.

Finally, we consider the effect of the sample size for a fixed number of sections with fixed section spacing (specifically, six sections with spacing $2R$, i.e. a slender specimen with form-factor $h/R = 10$). Fig. 5 displays the estimation errors before and after the removal of the polar caps from the Gaussian map (filtering), for sample sizes 200, 500, 800. It appears that the quality increases with the sample size, but very moderately.

In conclusion, we can offer the following metrological recommendation: The axis-estimation method is quite robust (not very sensitive to variations of measurement parameters) and is also quite effective, achieving a final error $< 10^{-3}$ (see Fig. 5). Since the ultimate objective is the evaluation of cylindricity, a denser sampling is recommended, consisting of a large number of sections and large number of samples per section.

4. Experimental results

We have applied our method to a number of examples. Two of these examples, CF2 and CF3, are taken from the paper by Carr and Ferreira [9], to which the reader is referred. Specimen CF2 is known to be near-vertical, but CF3 has a strongly nonvertical axis orientation. In [9] its axis position was supplied as an initial condition; our technique, on the contrary, provides an accurate axis estimate from the data set. In all cases, each data set is subjected to a rotation based on the axis estimate. Next, the repositioned data set is subjected to the hyperboloid algorithm. For each iteration we report the largest absolute value of the misalignment parameters (u, v, α, β).[2] The process terminates with the value 0 (i.e. a number smaller than the smallest number representable by the linear-programming algorithm), at which point cylindricity is reliably determined. Of course, the cylindricity results basically coincide with those of Carr and Ferreira, as was expected due to the analogy between the optimization algorithms. The results are reported in Table 1. The first column contains the specimen designations, the second, third, fourth, and fifth columns, the moduli of the maximum deviation, the sixth column, the cylinder radius, the seventh column, the cylindricity value, with the corresponding least-square estimate in the last column.

In addition, we also report results on two additional specimens, measured at Mahr Federal in Providence, RI. Each data set consists of three evenly spaced horizontal sections, each in turn comprising 4096 evenly spaced points.[3]

[2] See Appendix A.

[3] The original files are obtainable from G. Singh at Micromagnetics, Inc.

Specimen brXYZ is a rather coarse specimen, whereas dkXYZ is extremely fine. Each data set has been subjected to the same sequence of steps as the published Carr–Ferreira sets. The same excellent convergence behavior is observed, and the results are consistently far better than the corresponding least-square method estimates.

5. Remark

Experimental practice suggests that the hyperboloid method is adequately robust in the presence of initial misalignments, although pre-execution of axis alignment significantly reduces the number of iterations. The axis-alignment method, however, appears to be particularly relevant to three-dimensional straightness evaluation [11,12]. This problem has been frequently approached as a minimum-zone task, which makes axis-alignment appropriate.

In fact, the straightness problem can be briefly formulated as follows. A set of spatial measurements of a nominally straight edge has been obtained, and the results are presented as a collection of three-dimensional points. A natural measure of straightness is provided by the radius of the minimum-enclosing cylinder of the sample points. The hyperboloid method can be specialized to minimum-enclosing cylinder evaluations in a straightforward manner. Whereas, as noted earlier, in a cylindricity-evaluation setting approximate verticality of the specimens can be normally assumed, no such guarantee is available in straightness evaluations. Axis-alignment preprocessing completely obviates this shortcoming, and the subsequent hyperboloid processing provides an extremely accurate determination of the cylinder radius, which is a quantification of edge straightness.

Appendix A

In an (x, y, z) frame of reference a circular cylinder is defined by five parameters (α, β, u, v, R), where R is the radius, $(\alpha, \beta, 0)$ the point of intersection of the cylinder axis with the (x, y)-plane, and $(u, v, 1)$ a vector directed as the cylinder axis. Such cylinder will be denoted $C(\alpha, \beta, u, v, R)$.

The distance $d(x, y, z)$ (or d for short) of a point (x, y, z) in space from the cylinder axis is the length of the difference between the vector $w=(x-\alpha, y-\beta, z)$ and its projection on the cylinder axis (of length $(w\cdot a)/||a||$, where $a=(u, v, 1)$: see Fig. 6 for an illustration).

Thus, we have

$$||w||^2 = \left(\frac{w\cdot a}{||a||}\right)^2 + d^2$$

or

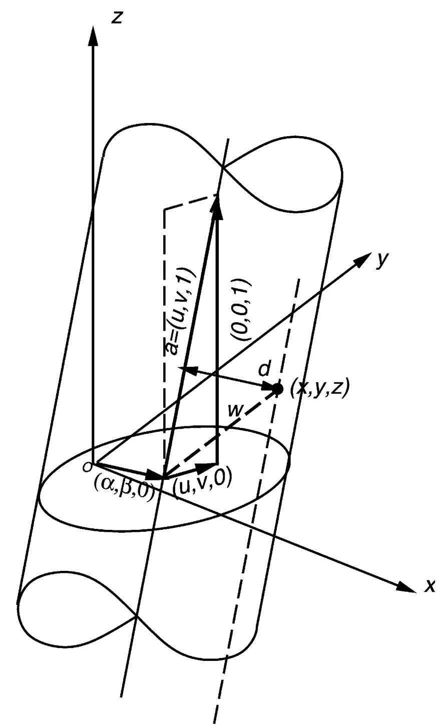

Fig. 6. Illustration of the frame of reference.

$$\begin{aligned} d^2 &= ||w||^2 - \left(\frac{w\cdot a}{||a||}\right)^2 = (x-\alpha)^2 + (y-\beta)^2 + z^2 \\ &\quad - \frac{1}{1+u^2+v^2}((x-\alpha)u + (y-\beta)v + z)^2 \\ &= \frac{1}{1+u^2+v^2}((x-\alpha)^2(1+v^2) + (y-\beta)^2(1+u^2) \\ &\quad + z^2(u^2+v^2) - 2z(x-\alpha)u - 2z(y-\beta)v \\ &\quad - 2(x-\alpha)(y-\beta)uv) \end{aligned}$$

that is,

$$\begin{aligned} &(x-\alpha-zu)^2 + (y-\beta-zv)^2 + ((x-\alpha)v - (y-\beta)u)^2 \\ &\quad = R^2(1+u^2+v^2) \qquad \text{(A.1)} \end{aligned}$$

is the equation of cylinder $C(\alpha, \beta, u, v, R)$.

By $C(\alpha, \beta, u, v)$ we denote the family of circular cylinders $\{C(\alpha, \beta, u, v, R) | R > 0\}$. Notice that the intersection of cylinder $C(\alpha, \beta, u, v, R)$ with a plane $z = c$ is an ellipse with center at $(x-\alpha-cu, y-\beta-cv, c)$, minor axis R, and major axis $R\sqrt{1+u^2+v^2}$ directed as vector $(u, v, 0)$.

If in Eq. (A.1), we retain only the terms that are at most linear in the parameters (α, β, u, v), we obtain the equation:

$$R^2 = x^2 + y^2 - 2uxz - 2vyz - 2\alpha x - 2\beta y \qquad \text{(A.2)}$$

which describes a one-sheet hyperboloid with axis defined by (α, β, u, v). Rewriting Eq. (A.2) as

$$\begin{aligned} &(x^2-\alpha-uz)^2 + (y-\beta-vz)^2 \\ &\quad = R^2 + (\alpha+uz)^2 + (\beta+vz)^2 \qquad \text{(A.3)} \end{aligned}$$

we see that the intersection of the hyperboloid with the plane $z = c$ is a circle with center at $(x - \alpha - cu, y - \beta - cv, c)$ (the same as for the elliptical sections of the cylinder $C(\alpha, \beta, u, v, R)$, which shows that the hyperboloid and $C(\alpha, \beta, u, v, R)$ are coaxial). We define as "radius" of the hyperboloid the radius $R' = \sqrt{R^2 + \alpha^2 + \beta^2}$ of its horizontal section at $z = 0$ and denote such hyperboloid $H(\alpha, \beta, u, v, R')$.

We now have the following important property:

Proposition 1.

$$\lim_{u,v,\alpha,\beta \to 0} H(\alpha, \beta, u, v, R') = C(\alpha, \beta, u, v, R)$$

Proof. Indeed, for $u = 0$, $v = 0$, $\alpha = 0$, and $\beta = 0$ both Eqs. (A.1) and (A.2) become

$$R^2 = x^2 + y^2$$

which is the equation of a circular cylinder with vertical axis passing by the origin. □

Definition 1. A zone cylinder $\mathcal{C}(\alpha, \beta, u, v, R + \tau, R + \eta)$ is a region of space contained between two coaxial cylinders $C(\alpha, \beta, u, v, R + \tau)$ (inner) and $C(\alpha, \beta, u, v, R + \eta)$ (outer).

Definition 2. A zone hyperboloid $\mathcal{H}(\alpha, \beta, u, v, R+\tau, R+\eta)$ is a region of space contained between two coaxial hyperboloids $H(\alpha, \beta, u, v, R + \tau)$ (inner) and $H(\alpha, \beta, u, v, R + \eta)$ (outer).

In both cases $\delta = \eta - \tau$ is called the "radial separation."

Consider now a finite set S of points (x_i, y_i, z_i) with $-h \le z_i \le h$. Typically S is a set of measurements of points on a nominally cylindrical surface in a metrological application. Let $\mathcal{H}(\alpha, \beta, u, v, R+\tau, R+\eta)$ be the zone hyperboloid containing S with the minimal value of $\eta - \tau = \delta$ among all the choices of $\alpha, \beta, u, v, \eta$ and τ. We call $\mathcal{H}(\alpha, \beta, u, v, R+\tau, R+\eta)$ the *minimal zone hyperboloid*.

Such minimal zone hyperboloid can be constructed as follows, under the assumption that the nominal radius R of the cylindrical surface is given. Each point of S must satisfy the inequalities

$$x_i^2 + y_i^2 - 2ux_iz_i - 2vy_iz_i - 2\alpha x_i - 2\beta y_i \le R^2 + 2R\eta \approx (R + \eta)^2$$

$$x_i^2 + y_i^2 - 2ux_iz_i - 2vy_iz_i - 2\alpha x_i - 2\beta y_i \ge R^2 + 2R\tau \approx (R + \tau)^2$$

Thus, the calculation of $(\alpha, \beta, u, v, \tau, \eta)$ is a linear-programming problem in six-dimensions with $2|S|$ constraints and objective function $\eta - \tau$.

The parameters $\{\alpha, \beta, u, v\}$ express the misalignment of the data point set. The misalignment will be reduced by the rigid transformation

$$\begin{vmatrix} x'_i \\ y'_i \\ z'_i \end{vmatrix} = \begin{vmatrix} \frac{u}{AB} & \frac{v}{AB} & -\frac{A}{B} \\ -\frac{v}{A} & \frac{u}{A} & 0 \\ \frac{u}{B} & \frac{v}{B} & \frac{1}{B} \end{vmatrix} \times \begin{vmatrix} x_i - \alpha \\ y_i - \beta \\ z_i \end{vmatrix}$$

where $A = \sqrt{u^2 + v^2}$ and $B = \sqrt{1 + u^2 + v^2}$. The optimization step is then iteratively applied to the set (x'_i, y'_i, z'_i), with the corrected radius $R' = R + (\eta + \tau)/2$.

We now wish to construct a minimal zone cylinder which is guaranteed to contain the target point set S and whose radial separation, therefore, provides (an upper-bound to) the cylindricity of S. We start from the computed optimal zone hyperboloid $\mathcal{H}(\alpha, \beta, u, v, R+\tau, R+\eta)$, whose inner and outer hyperboloids have respectively the following equations:

$$(R + \tau)^2 = x^2 + y^2 - 2uxz - 2vyz - 2\alpha x - 2\beta y$$

$$(R + \eta)^2 = x^2 + y^2 - 2uxz - 2vyz - 2\alpha x - 2\beta y$$

Our zone cylinder will be coaxial with the minimal zone hyperboloid $\mathcal{H}(\alpha, \beta, u, v, R + \tau, R + \eta)$, and therefore will be denoted $\mathcal{C}(\alpha, \beta, u, v, R^* + \tau^*, R^* + \eta^*)$.

It is clear that the desired $\mathcal{C}(\alpha, \beta, u, v, R^* + \tau^*, R^* + \eta^*)$ is the zone cylinder of the chosen type with minimal $\delta^* = \eta^* - \tau^*$ that contains the intersection of $\mathcal{H}(\alpha, \beta, u, v, R+\tau, R+\eta)$ with the z-range $[-h_{\max}, h_{\max}]$, where $h_{\max} = \max_{i=1}^{n}|z_i|$ (as justified by the definition of the measurement point set). The reader is referred to Fig. 7 for an effective illustration of the relationship between a zone hyperboloid and its associated zone cylinder.

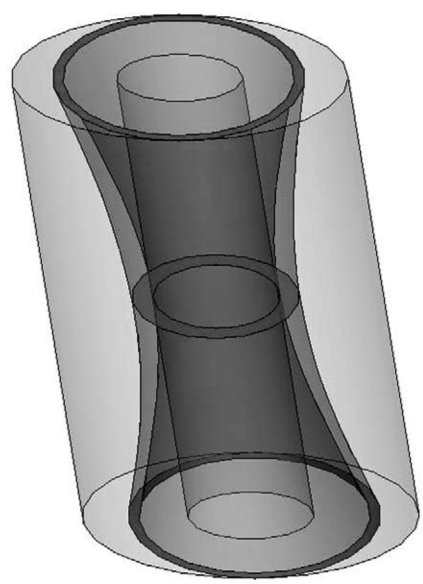

Fig. 7. Illustration of the relationship between a zone hyperboloid and its associated zone cylinder.

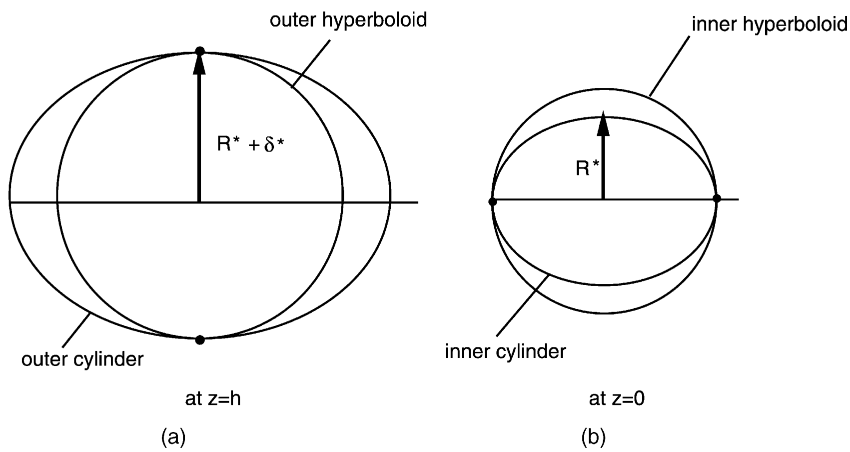

Fig. 8. Illustration of horizontal sections of the zone hyperboloid and its associated zone cylinder.

A complete characterization of $\mathcal{C}(\alpha, \beta, u, v, R^*+\tau^*, R^*+\eta^*)$ is provided by the intersections with the planes $z = 0$ and $z = h$. We let $R' = R + \tau$, $R'' = R^* + \tau^*$, and $\delta = \eta - \tau$.

In plane $z = 0$ (see Fig. 8b), the intersection of the inner cylinder must be contained within the intersection of the inner hyperboloid, and tangent to it. As noted earlier, the former is an ellipse with minor semi-axis R'' and major semi-axis $R''\sqrt{1+u^2+v^2}$, and the latter is a circle of radius $\sqrt{R'^2+\alpha^2+\beta^2}$. Clearly, the circle radius is equal to the major semi-axis, i.e.

$$R'\sqrt{1+u^2+v^2} = \sqrt{R'^2+\alpha^2+\beta^2}$$

from which we obtain

$$R'' = R'\sqrt{\frac{1+(\alpha^2+\beta^2)/R'^2}{1+u^2+v^2}}$$

In plane $z = h$ (see Fig. 8a), the intersection of the outer cylinder (an ellipse) must contain, and be tangent to, the intersection of the outer hyperboloid (a circle). This implies that the minor semi-axis of the ellipse equals the radius of the circle. By Eq. (A.1), the latter has value

$$\sqrt{(R'+\delta)^2+(uh+\alpha)^2+(vh+\beta)^2}$$

Since the radius of the outer cylinder equals the ellipse's minor semi-axis, it follows that

$$R''+\delta^* = \sqrt{(R'+\delta)^2+(uh+\alpha)^2+(vh+\beta)^2}$$

Therefore, we conclude that

$$\delta^* = (R'+\delta)\sqrt{1+\frac{(uh+\alpha)^2+(vh+\beta)^2}{(R'+\delta)^2}} - R'\sqrt{\frac{1+(\alpha^2+\beta^2)/R'^2}{1+u^2+v^2}}$$

If we now neglect $(\alpha^2+\beta^2)/R'^2$ with respect to 1, and use the standard inequalities

$$\sqrt{1+x} \le 1+\frac{x}{2}, \qquad \frac{1}{\sqrt{1+x}} \ge 1-\frac{x}{2}$$

we obtain:

$$\delta^* \le \delta + \frac{(uh+\alpha)^2+(vh+\beta)^2}{2(R'+\delta)} + R'\frac{u^2+v^2}{2}$$

Finally, we may neglect δ with respect to R', substitute R' for h where appropriate since $R' \approx h$, and define $\epsilon = \max\{\alpha, \beta, uh, vh\}$, obtaining

$$\delta^* \le \delta + \frac{5\epsilon^2}{R'}$$

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