

## Listing All Plane Graphs

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### Abstract

In this paper we give a simple algorithm to generate all connected rooted plane graphs with at most  $m$  edges. A “rooted” plane graph is a plane graph with one designated (directed) edge on the outer face. The algorithm uses  $O(m)$  space and generates such graphs in  $O(1)$  time per graph on average without duplications. The algorithm does not output the entire graph but the difference from the previous graph. By modifying the algorithm we can generate all connected (non-rooted) plane graphs with at most  $m$  edges in  $O(m^3)$  time per graph.

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Generating all graphs with some property without duplications has many applications, including unbiased statistical analysis [9]. A lot of algorithms to solve these problems are known such as [1, 2, 8, 9, 10, 15]. See textbooks [5, 6, 7, 13, 14].

In this paper we wish to generate all connected “rooted” plane graphs, which will be defined precisely in Section 2, with at most  $m$  edges. Such graphs play an important role in many algorithms, including graph drawing algorithms such as [3, 4, 12]. Using our generation algorithm one can test an implementation of a plane graph drawing algorithm. Our algorithm constructs the complete list of rooted plane graphs with at most  $m$  edges.

To solve these all-graph-generating problems some types of algorithms are known.

Classical method algorithms [5, p.57] first generate all the graphs with a given property allowing duplications, but output only if the graph has not been output yet. Thus this method requires quite a huge space to store a list of graphs that have already been output. Furthermore, checking whether each graph has already been output requires a lot of time.

Orderly method algorithms [5, p.57] need not store the list, since they output a graph only if it is a “canonical” representative of each isomorphism class.

Reverse search method algorithms [1] also need not store the list. The idea is to implicitly define a connected graph  $H$  such that the vertices of  $H$  correspond to the graphs with the given property, and the edges of  $H$  correspond to some relation between the graphs. By traversing an implicitly defined spanning tree of  $H$ , one can find all the vertices of  $H$ , which correspond to all the graphs with the given property without duplication.

The main idea of our algorithms is that for some problems (biconnected triangulations [8], and triconnected triangulations [10]) we can define a tree (not a general graph) as the graph  $H$  of the reverse search method. Thus our algorithms do not need to find a spanning tree of  $H$ , since  $H$  itself is a tree. With some other ideas we give the following two simple but efficient algorithms.

Our first algorithm generates all simple connected rooted plane graphs with at most  $m$  ( $m > 0$ ) edges. *Simple* means there is neither self loops nor multiple edges. A *rooted* plane graph means a plane graph with one designated “root” edge on the outer face. Its precise definition is given in the next section. For instance there are nine simple connected rooted plane graphs with at most three edges, as shown in Figure 1(a). The root edges are depicted by thick grey lines. However, there are only five simple connected (non-rooted) plane graphs with at most three edges. See Figure 1(b). The algorithm uses  $O(m)$  space and runs in  $O(g(m))$  time, where  $g(m)$  is the number of nonisomorphic connected rooted plane graphs with at most  $m$  edges. The algorithm generates each graph in  $O(1)$  time on average without duplications. The algorithm does not output the entire graph but the difference from the previous graph.

By modifying the algorithm we can generate all connected (non-rooted) plane graphs with at most  $m$  edges in  $O(m^3)$  time per graph.

The rest of the paper is organized as follows. Section 2 gives some definitions. Section 3 shows a tree structure among connected rooted plane graphs. Section

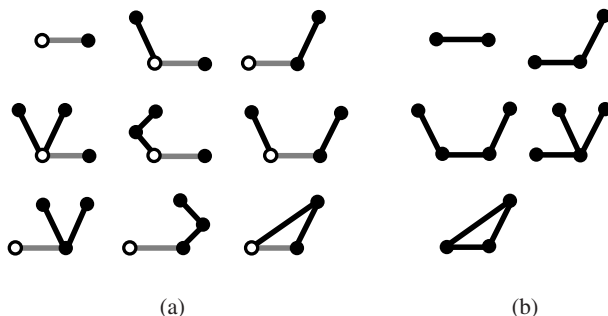


Figure 1: (a) Connected rooted plane graphs, and (b) connected (non-rooted) plane graphs.

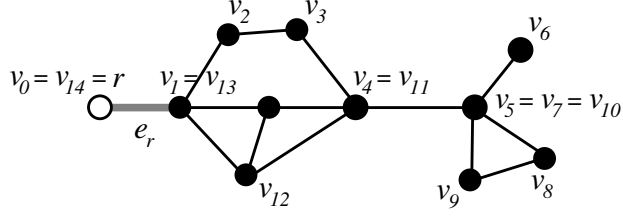
4 presents our first algorithm to generate all connected rooted plane graphs. Then, by modifying the algorithm we give an algorithm to generate all connected (non-rooted) plane graphs. Section 5 analyzes the running time of our algorithm. Finally Section 6 is a conclusion.

## 1 Preliminaries

In this section we give some definitions.

Let  $G$  be a connected graph with  $m$  edges. In this paper all graphs are simple, so there is neither self loops nor multiple edges. An edge connecting vertices  $u$  and  $w$  is denoted by  $(u, w)$ . The *degree* of a vertex  $v$  is the number of neighbors of  $v$  in  $G$ .

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A *plane graph* is a planar graph with a fixed planar embedding. A plane graph divides the plane into connected regions called *faces*. The unbounded face is called *the outer face*, and other faces are called *inner faces*. We regard *the contour* of a face as the clockwise cycle formed by the vertices on the boundary of the face. We denote the contour of the outer face of plane graph  $G$  by  $C_o(G)$ . For instance, in Figure 2,  $C_o(G) = v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7(= v_5), v_8, v_9, v_{10}(= v_5), v_{11}(= v_4), v_{12}, v_{13}(= v_1), v_{14}(= v_0)$ . Note that a vertex may appear several times on  $C_o(G)$ . We say each  $v_i$  on  $C_o(G)$  is an *appearance* of a vertex. For instance  $v_5, v_7$  and  $v_{10}$  are the appearances of the same vertex  $v_5 = v_7 = v_{10}$ . A *rooted plane graph* is a plane graph with one designated edge  $e_r = (v_l, v_r)$  on  $C_o(G)$ . We assume that  $v_l$  succeeds  $v_r$  on  $C_o(G)$ ,  $v_l$  appears in the head on  $C_o(G)$ , and  $v_r$  appears at the end on  $C_o(G)$  except  $v_l$ . The designated edge is called *the root edge*, and vertex  $v_l$  is called *the root vertex*. Note that a rooted plane graph has one or more edges. The root edges and root vertices are denoted by a grey lines and white circles in all figures, respectively. From now on we write  $r$  for the root vertex.

Figure 2: A connected rooted plane graph  $G$ .

## 2 The Removing Sequence and the Family Tree

Let  $S_m$  be the set of all connected rooted plane graphs with at most  $m$  edges. In this section we explain a tree structure relating the graphs in  $S_m$ .

Let  $G$  be a connected rooted plane graph with two or more edges. Let  $e_r = (v_{k-1}, v_0)$  be the root edge of  $G$  and  $C_o(G) = v_0(=r), v_1, v_2, \dots, v_{k-1}, v_0(=r)$ . Note that  $v_0$  succeeds  $v_{k-1}$  on  $C_o(G)$ .

We classify the edges on  $C_o(G)$  into three types as follows. If an edge  $e$  on  $C_o(G)$  is included in a cycle of  $G$  then  $e$  is a *cycle edge*. Otherwise, if at least one vertex of  $e$  has degree 1 then  $e$  is a *pendant*. Otherwise  $e$  is a *bridge*. We can observe if we remove a bridge from  $G$  then the resulting graph is disconnected. For instance, in Figure 2, the edge  $(v_2, v_3)$  is a cycle edge,  $(v_5, v_6)$  is a pendant, and  $(v_4, v_5)$  is a bridge.

An edge  $e \neq e_r$  on  $C_o(G)$  is *removable* if and only if  $e$  is either a pendant or a cycle edge. If  $e$  is a removable edge then after removing  $e$  from  $G$  the remaining edges induce a connected graph. Note that if  $e$  is a cycle edge, removing  $e$ , we have the resultant graph, on the other hand, if  $e$  is a pendant, removing the isolated vertex after removing  $e$ , we have the resultant graph. Thus the resultant graph is connected in both cases. Since  $G$  is a rooted plane graph, the resultant graph after removing a removable edge is also a rooted plane graph with the same root edge.

We have the following lemma.

**Lemma 1** *Every connected rooted plane graph with two or more edges has at least one removable edge.*

**Proof:** Let  $G$  be a connected rooted plane graph with two or more edges, with the root edge  $e_r = (v_{k-1}, v_0)$ , and  $C_o(G) = v_0(=r), v_1, v_2, \dots, v_{k-1}, v_0$ . Let  $e$  be the first edge distinct from  $e_r$  on  $C_o(G)$ . Now  $e$  must be one of the three types, that is, a bridge, a pendant or a cycle edge. If  $e$  is a pendant or a cycle edge, it is removable, and we are done. Otherwise  $e$  is a bridge, then on  $C_o(G)$  the next edge of  $e$  is either a pendant, a bridge or a cycle edge. By repeating this procedure we can find at least one pendant or cycle edge, which is removable.  $\square$

If  $e_a = (v_{a-1}, v_a)$ ,  $a > 1$ , is removable but none of  $(v_0, v_1), (v_1, v_2), \dots, (v_{a-2},$

$v_{a-1}$ ) is removable, then  $e_a$  is called *the first removable edge* of  $G$ . We can observe that if  $e_a$  is the first removable edge then each of  $(v_0, v_1), (v_1, v_2), \dots, (v_{a-2}, v_{a-1})$  is a bridge or the root edge. (So they are not removable.)

For each graph  $G$  in  $S_m$  except  $K_2$ , if we remove the first removable edge then the resulting edge-induced graph, denoted by  $P(G)$ , is also a graph in  $S_m$  having one less edge. Thus we can define the unique graph  $P(G)$  in  $S_m$  for each  $G$  in  $S_m$  except  $K_2$ . We say  $G$  is a *child* graph of  $P(G)$ .

Given a graph  $G$  in  $S_m$ , by repeatedly removing the first removable edge, we can have the unique sequence  $G, P(G), P(P(G)), \dots$  of graphs in  $S_m$  which eventually ends with  $K_2$ . By merging those sequences we can have *the family tree*  $T_m$  of  $S_m$  such that the vertices of  $T_m$  correspond to the graphs in  $S_m$ , and each edge corresponds to each relation between some  $G$  and  $P(G)$ . For instance  $T_4$  is shown in Figure 3, in which each first removable edge is depicted by a thick black line. We call the vertex in  $T_m$  corresponding to  $K_2$  *the root* of  $T_m$ .

### 3 Algorithms

The outline of our algorithm is as follows. By traversing  $T_m$  we can generate all vertices of  $T_m$  and corresponding all connected rooted plane graphs in  $S_m$ . If we can find all child graphs of the current graph in  $T_m$ , then with a recursive manner we can traverse  $T_m$ . Thus we only need to design an algorithm to generate all child graphs of a given current graph in  $S_m$ . Similar technique is used in [8, 10] to efficiently generate some classes of graphs (biconnected triangulations and triconnected triangulations) but based on different family trees.

Let  $e_r$  be the root edge. Let  $C_o(G) = v_0(= r), v_1, \dots, v_{k-1}, v_0(= r)$ , and  $(v_{a-1}, v_a)$  be the first removable edge of  $G$ . Note that  $k$  is the number of appearances of the vertices on the contour of the outer face. Since  $K_2$  has no removable edge, for convenience, we regard  $e_1 = (v_0, v_1)$  as the first removable edge for  $K_2$ . We denote by  $G(i)$ ,  $0 \leq i < k$ , the rooted plane graph obtained from  $G$  by adding a new pendant at  $v_i$ , and by  $G(i, j)$ ,  $0 \leq i < j < k$ , the rooted plane graph obtained from  $G$  by adding a new cycle edge connecting  $v_i$  and  $v_j$  on the outer face of  $G$ , as shown in Figure 4. We can observe that each child of  $G$  is either  $G(i)$  or  $G(i, j)$  for some  $i$  and  $j$ , and  $G(i)$  or  $G(i, j)$  is a child graph of  $G$  if and only if the newly added edge of  $G(i)$  or  $G(i, j)$  is the first removable edge.

If  $(v_{a-1}, v_a)$  is the first removable edge of  $G$ , then edges  $(v_0, v_1), (v_1, v_2), \dots, (v_{a-2}, v_{a-1})$  are bridges or the root edge, and vertices  $v_0, v_1, v_2, \dots, v_a$  form a path on  $C_o(G)$ . We call this path *the critical path* of  $G$  and denote it  $P_c(G)$ . For instance, in Figure 2,  $P_c(G) = (v_0, v_1, v_2)$ .

Now we are going to find all child graphs of  $G$ . We have the following two cases to consider. Let  $b(i)$  be the largest integer satisfying  $v_i = v_{b(i)}$ . Thus  $v_{b(i)}$  is the last appearance of  $v_i$  on  $C_o(G)$ .

**Case 1:** The first removable edge  $(v_{a-1}, v_a)$  of  $G$  is a pendant (including the special case when  $G$  is  $K_2$ ).

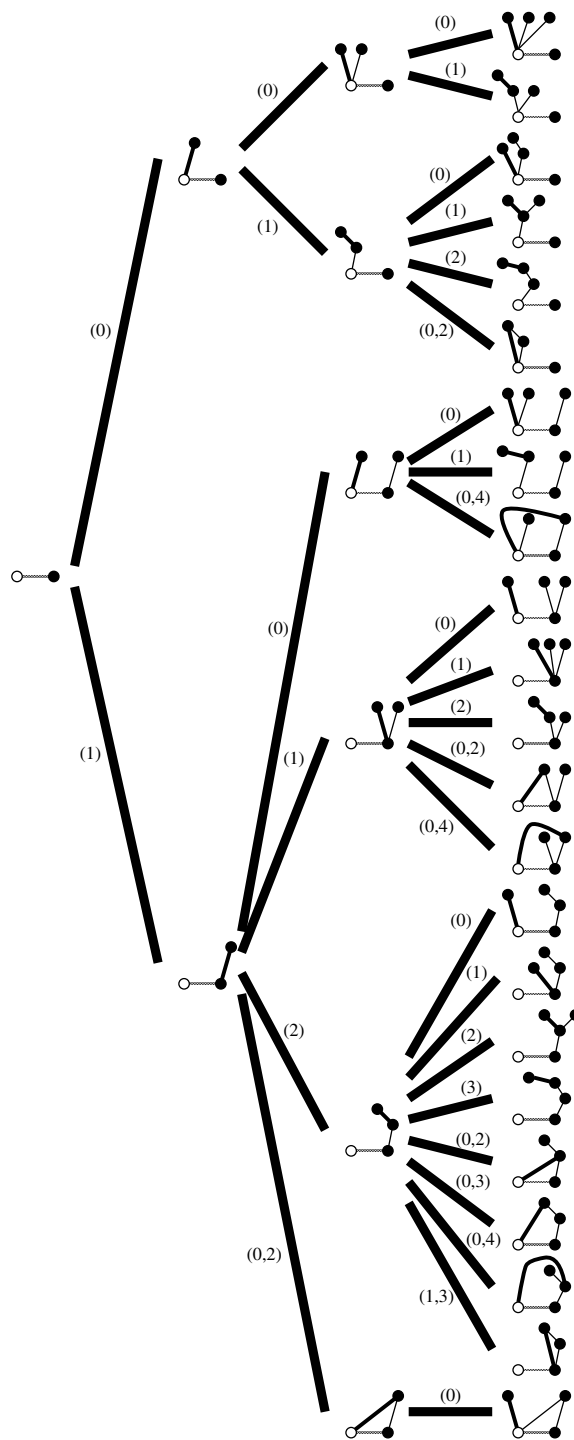


Figure 3: The family tree  $T_4$ .

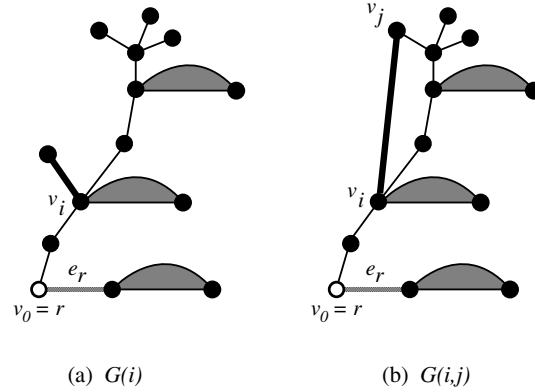


Figure 4: Illustration for (a)  $G(i)$  and (b)  $G(i, j)$ .

Consider graphs  $G(i)$ ,  $0 \leq i \leq k$ . For each  $i$ ,  $0 \leq i \leq a$ , the newly added edge in  $G(i)$  is the first removable edge of  $G(i)$ , thus  $P(G(i)) = G$ . For each  $i$ ,  $a < i < k$ ,  $(v_{a-1}, v_a)$  is still the first removable edge of  $G(i)$ , so  $P(G(i)) \neq G$ .

Then consider graphs  $G(i, j)$ ,  $0 \leq i < j < k$ . For each  $i$  and  $j$ , ( $i < j$ ), if (1)  $v_i \neq v_j$ , (2)  $0 \leq i \leq a - 1$ , (3)  $(v_i, v_j)$  is not an edge of  $G$ , and (4)  $j < b(i)$ , then the newly added edge in  $G(i, j)$  is the first removable edge of  $G(i, j)$ , thus  $P(G(i, j)) = G$ . Note that if  $v_i = v_j$  edge  $(v_i, v_j)$  is a self loop (even if  $i \neq j$ ), and so  $G(i, j)$  is not simple. Also if  $G$  already has the edge  $(v_i, v_j)$  then  $G(i, j)$  has a multiple edge, and so  $G(i, j)$  is not simple. If  $i \geq a$ , then the newly added edge in  $G(i, j)$  is not the first removable edge of  $G(i, j)$ , since  $(v_{a-1}, v_a)$  is still removable, thus  $P(G(i, j)) \neq G$ . Otherwise,  $0 \leq i \leq a - 1$  and  $j > b(i)$  holds. Let  $s(j)$  be the smallest integer satisfying  $v_j = v_{s(j)}$ . Note that  $s(j) < i$  holds. Now edges  $(v_{s(j)}, v_{s(j)+1}), (v_{s(j)+1}, v_{s(j)+2}), \dots, (v_{i-1}, v_i)$  are cycle edges in  $G(i, j)$  since adding edge  $(i, j)$  to them completes a cycle. Thus the newly added edge  $(i, j)$  is not the first removable edge of  $G(i, j)$  so  $P(G(i, j)) \neq G$ .

**Case 2:** The first removable edge  $(v_{a-1}, v_a)$  of  $G$  is a cycle edge.

Consider graphs  $G(i)$ ,  $0 \leq i < k$ . For each  $i$ ,  $0 \leq i \leq a - 1$ , the newly added edge in  $G(i)$  is the first removable edge of  $G(i)$ , so  $P(G(i)) = G$ . For each  $i$ ,  $a \leq i < k$ ,  $(v_{a-1}, v_a)$  is still the first removable edge of  $G(i)$ , so  $P(G(i)) \neq G$ .

Then consider graphs  $G(i, j)$ ,  $0 \leq i < j < k$ . For each  $i$  and  $j$ , ( $i < j$ ), if (1)  $v_i \neq v_j$ , (2)  $0 \leq i \leq a - 1$ , (3)  $(v_i, v_j)$  is not an edge of  $G$ , and (4)  $j < b(i)$ , then the newly added edge in  $G(i, j)$  is the first removable edge of  $G(i, j)$ , thus  $P(G(i, j)) = G$ . Note that if  $v_i = v_j$  edge  $(v_i, v_j)$  is a self loop (even if  $i \neq j$ ), and so  $G(i, j)$  is not simple. Also if  $G$  already has the edge  $(v_i, v_j)$  then  $G(i, j)$  has a multiple edge, and so  $G(i, j)$  is not simple. If  $i \geq a$ , then the newly added edge in  $G(i, j)$  is not the first removable edge of  $G(i, j)$ , since  $(v_{a-1}, v_a)$  is still removable, thus  $P(G(i, j)) \neq G$ . Otherwise,  $0 \leq i \leq a - 1$  and  $j > b(i)$  holds. Let  $s(j)$  be the smallest integer satisfying  $v_j = v_{s(j)}$ . Note that  $s(j) < i$  holds. Now edges  $(v_{s(j)}, v_{s(j)+1}), (v_{s(j)+1}, v_{s(j)+2}), \dots, (v_{i-1}, v_i)$  are cycle edges

in  $G(i, j)$  since adding edge  $(i, j)$  to them completes a cycle. Thus the newly added edge  $(i, j)$  is not the first removable edge of  $G(i, j)$  so  $P(G(i, j)) \neq G$ .

Based on the case analysis above we can find all child graphs of any given graph in  $S_m$ . If  $G$  has  $l$  child graphs, then we can find them in  $O(l)$  time with a suitable data structure, which will be described in Section 5. This is an intuitive reason why our algorithm generates each graph in  $O(1)$  time per graph on average.

Recursively repeating this process from the root of  $T_m$  corresponding to  $K_2$  we can traverse  $T_m$  without constructing the whole part of  $T_m$  at once. During the traversal of  $T_m$ , we assign a label  $(i)$  or  $(i, j)$  to each edge connecting  $G$  and either  $G(i)$  or  $G(i, j)$  in  $T_m$ , as shown in Figure 3. Each label denotes how to add a new edge to  $G$  to generate a child graph  $G(i)$  or  $G(i, j)$ , and each sequence of labels on a path starting from the root specifies a graph in  $S_m$ . For instance, the sequence  $(1)(0, 2)(0)$  specifies the right-bottom graph in Figure 3. During our algorithm we will maintain these labels only on the path from the root to the “current” vertex of  $T_m$ , because those labels carry enough information to generate the “current” graph. To generate the next graph, we need to maintain more information only for the graphs on the “current” path, which has length at most  $m$ , and each graph can be represented as a constant size of difference from the preceding one. This is an intuitive reason why our algorithm uses only  $O(m)$  space, while the number of graphs may not be bounded by a polynomial in  $m$ .

Our algorithm is as follows.

**Procedure find-all-child-graphs( $G$ )**

**begin**

- 01 Output  $G$  {Output the difference from the previous graph.}
- 02 Assume  $(v_{a-1}, v_a)$  is the first removable edge of  $G$ .
- 03 **if**  $G$  has exactly  $m$  edges **then return**
- 04 **for**  $i = 0$  **to**  $a - 1$  {Cases 1 and 2}
- 05 **find-all-child-graphs**( $G(i)$ )
- 06 **if**  $(v_{a-1}, v_a)$  is a pendant **then** {Case 1}
- 07 **find-all-child-graphs**( $G(a)$ )
- 08 **for**  $i = 0$  **to**  $a - 1$  {Cases 1 and 2}
- 09 **for**  $j = i + 2$  **to**  $b(i) - 1$
- 10 **if**  $v_i \neq v_j$  and  $(v_i, v_j)$  is not an edge of  $G$  **then**
- 11 **find-all-child-graphs**( $G(i, j)$ )

**end**

**Algorithm find-all-graphs( $T_m$ )**

**begin**

- 1 Output  $K_2$
- 2  $G = K_2$

```

3  find-all-child-graphs( $G(0)$ )
4  find-all-child-graphs( $G(1)$ )
end

```

We have the following theorem. The proof is given in Section 5.

**Theorem 1** *The algorithm uses  $O(m)$  space and runs in  $O(g(m))$  time, where  $g(m)$  is the number of nonisomorphic connected rooted plane graphs with at most  $m$  edges.*

We can modify our algorithm so that it outputs all connected (non-rooted) plane graphs with at most  $m$  edges, as follows. At each vertex  $v$  of the family tree  $T_m$ , the graph  $G$  corresponding to  $v$  is checked whether the sequence of labels of  $G$  (with the root edge) is the lexicographically first one among the  $k$  sequences of labels of  $G$  for the  $k$  choices of the root edge on  $C_o(G)$ , and only if so,  $G$  is output. Thus we can output only the canonical representative of each isomorphism class. A similar method has appeared in [8, 10].

**Lemma 2** *The algorithm uses  $O(m)$  space and runs in  $O(m^3 \cdot h(m))$  time, where  $h(m)$  is the number of nonisomorphic connected (non-rooted) plane graphs with at most  $m$  edges.*

**Proof:** For each graph corresponding to a vertex of  $T_m$  we construct  $k \leq m$  of sequences of labels corresponding to the  $k$  choices for the root edge on  $C_o(G)$  in  $O(m)$  time for each sequence, and find the lexicographically first one in  $O(km)$  time. And for each output graph, our tree may contain  $k$  of isomorphic ones corresponding to the  $k$  choices for the root edge. Thus the algorithm runs in  $O(k^2m \cdot h(m)) = O(m^3 \cdot h(m))$  time. The algorithm clearly uses  $O(m)$  space.  $\square$

## 4 Proof of Theorem 1

In this section we give a proof of Theorem 1, that is if  $G$  has  $l$  child graphs how we can find them in  $O(l)$  time.

Given a connected rooted plane graph  $G$  in  $S_m$  with at most  $m - 1$  edges, we are going to find all child graphs of  $G$  by algorithm **find-all-child-graphs**. Let  $(v_{k-1}, v_0)$  be the root edge of  $G$ ,  $C_o(G) = v_0(= r), v_1, v_2, \dots, v_{k-1}, v_0(= r)$ , and  $(v_{a-1}, v_a)$  be the first removable edge of  $G$ .

If  $G$  has  $l$  child graphs of type  $G(i)$ , by only maintaining the critical path  $v_0, v_1, \dots, v_a$ , we can find such child graphs in  $O(l)$  time. See lines 04–07 of **find-all-child-graphs**.

On the other hand, if  $G$  has  $l'$  child graphs of type  $G(i, j)$ , we need to maintain a slightly complicated data structure to find all such child graphs in  $O(l')$  time. Note that if either (1)  $v_i = v_j$ , or (2)  $G$  has an edge  $(v_i, v_j)$ , then  $G(i, j)$  is not simple and  $G(i, j)$  is not a child graph of  $G$ , so we need to efficiently skip such  $j$ 's at line 10. For each of the other  $j$ 's, we need to generate  $G(i, j)$ , since those are child graphs of  $G$ .

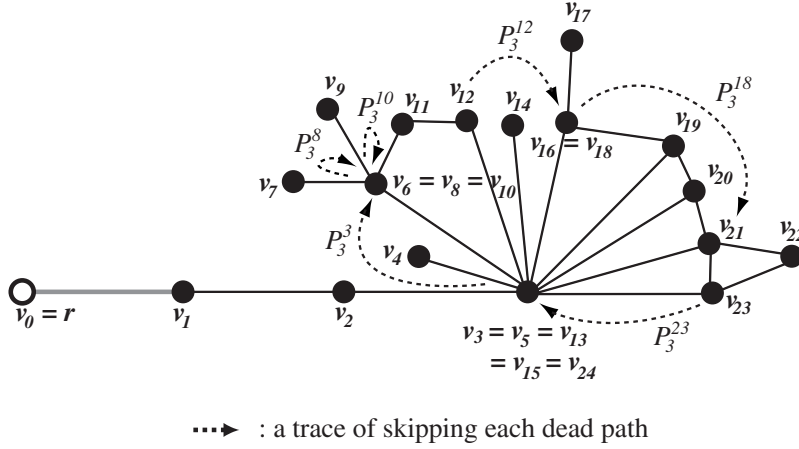


Figure 5: An illustration for the zombie list of  $v_3$ .

Our idea is as follows. Let  $v_i$  be an appearance of a vertex on the critical path of  $G$ . We say that an appearance  $v_j$  on  $C_o(G)$  is *dead* with respect to  $v_i$  if either (1)  $v_i = v_j$ , or (2)  $G$  has an edge  $(v_i, v_j)$ . To skip dead appearances efficiently, for each vertex  $v_i$  on the critical path, we maintain a list of successive dead appearances with respect to  $v_i$ , which allows us to skip each run of successive dead appearances in  $O(1)$  time. After each time skipping successive dead appearances we can always generate a child graph of  $G$  corresponding to the next “non-dead” appearance. Thus  $l'$  child graphs of type  $G(i, j)$  can be generated in  $O(l')$  time. The details are as follows.

Let  $v_{a(i)}$  and  $v_{b(i)}$  be the first and last appearances of  $v_i$  on  $C_o(G)$ . Let  $P_i$  be the subpath from  $v_{a(i)}$  to  $v_{b(i)}$  on  $C_o(G)$ . A maximal subpath  $P_i^c$  of  $P_i$  is called a *dead path* of  $v_i$  if all appearances  $v_c, v_{c+1}, \dots$  on  $P_i^c$  are dead with respect to  $v_i$ . For example, the graph in Figure 5 has 6 dead paths of  $v_3$ :  $P_3^3 = (v_3, v_4, v_5, v_6)$ ,  $P_3^8 = (v_8)$ ,  $P_3^{10} = (v_{10})$ ,  $P_3^{12} = (v_{12}, v_{13}, v_{14}, v_{15}, v_{16})$ ,  $P_3^{18} = (v_{18}, v_{19}, v_{20}, v_{21})$  and  $P_3^{23} = (v_{23}, v_{24})$ . They appear on  $C_o(G)$  in this order. For each  $v_i$  ( $0 \leq i \leq a-1$ ), we maintain all dead paths as a list, and we call the list as *the zombie list* of  $v_i$ . Using the zombie list we can skip each run of successive dead appearances in  $O(1)$  time. After each time we skip a dead path, we can always generate at least one child graph. Thus, we can generate each child graph of type  $G(i, j)$  in  $O(1)$  time.

Now we show how to prepare those data structures for each child graph.

Given a connected rooted plane graph  $G$  and the zombie list of each vertex on the critical path, we are going to generate all child graphs, and for each child graph we prepare the zombie list of each vertex on the new critical path by modifying the list for  $G$ .

We have the following two cases.

**Case 1:** Child graphs of type  $G(i)$ .

We have the following two cases.

**Case 1(a):**  $i = a$ .

The first removable edge of  $G$  is a pendant, since otherwise the first removable edge of  $G$  is a cycle edge and  $G(i)$  is not a child graph of  $G$ . Adding the new pendant at  $v_i$  on the critical path of  $G$  generates the critical path of  $G(i)$  with one more length. The zombie list of each  $v_l$ ,  $0 \leq l \leq a - 2$ , for  $G(i)$  is identical to the ones in  $G$ .

The zombie list of  $v_{a-1}$  for  $G(i)$  is derived by dividing the first dead path  $P$  of  $v_{a-1}$  in  $G$  as follows. Let  $P = (v_{a-1}, v_a, v'_1, v'_2, \dots)$  then we divide  $P$  into two dead paths  $P_1 = (v_{a-1}, v_a)$  and  $P_2 = (v_a, v_1, v_2, \dots)$ . Note that adding the new edge generates one more appearance of  $v_a$ . See an example in Figure 6(a). The dead path  $P_2^2$  in Figure 6(a) is divided into  $P_2^2$  and  $P_2^3$ . Other dead paths of  $v_{a-1}$  in  $G(i)$  are identical to the ones in  $G$ .

The zombie list of  $v_a$  consists of one dead path  $P = (v_a, v_x, v_a)$ , where  $v_x$  is the other end vertex of the new edge.

Thus we can modify the zombie list of each vertex on the critical path in  $O(1)$  time.

**Case 1(b):** Otherwise.

The critical path of  $G(i)$  is  $v_0, v_1, \dots, v_i, v_x$ , where  $v_x$  is the other end vertex of the new edge.

The zombie list of each  $v_l$ ,  $0 \leq l \leq i - 1$  for  $G(i)$  is identical to the zombie list for  $G$ .

The zombie list of  $v_i$  is derived by appending  $(v_i, v_x)$  as the prefix to the first dead path of  $v_i$ . See an example in Figure 6(b). By appending  $(v_2, v_x)$  into the dead path  $P_2^2$  of  $v_2$  in  $G$ , the dead path  $P_2^2$  of  $G(i)$  is derived. Note that the other dead path of  $v_i$  in  $G(i)$  is identical to the ones in  $G$ .

Thus we can modify the zombie list of each vertex on the critical path in  $O(1)$  time.

**Case 2:** Child graphs of type  $G(i, j)$ .

The critical path of  $G(i, j)$  is  $v_0, v_1, \dots, v_i, v_j$ .

Note that  $v_{i+1}, v_{i+2}, \dots, v_{j-1}$  are not on  $C_o(G(i, j))$ . So we need not maintain the zombie lists of those. Also each  $v_{j+1}, v_{j+2}, \dots, v_a$  are not on the critical path of  $G(i, j)$ . So we need not maintain the zombie lists of those.

The zombie list of each  $v_l$ ,  $0 \leq l \leq i - 1$ , is identical to the zombie list for  $G(i)$ .

The zombie list of  $v_i$  is derived by removing dead paths of  $v_i$  up to  $v_j$  on  $C_o(G)$ . If  $v_{j+1}$  is dead with respect to  $v_i$  in  $G$ , then appending  $(v_i, v_j)$  into the dead path  $P_i^{j+1} = (v_{j+1}, v_{j+2}, \dots)$  generates the zombie list of  $v_i$  for  $G(i, j)$ . See an example in Figure 6(c). By appending  $(v_2, v_5)$  into the dead path  $P_2^5$  of  $v_2$  in  $G$ , the dead path  $P_2^2$  of  $v_2$  in  $G(i, j)$  is derived. Otherwise if  $v_{j+1}$  is not dead then we append a new dead path  $P_i^i = (v_i, v_j)$  into the zombie list of  $v_i$ . Other dead paths remain as they are.

Thus we can modify the zombie list of each vertex on the critical path in  $O(1)$  time.

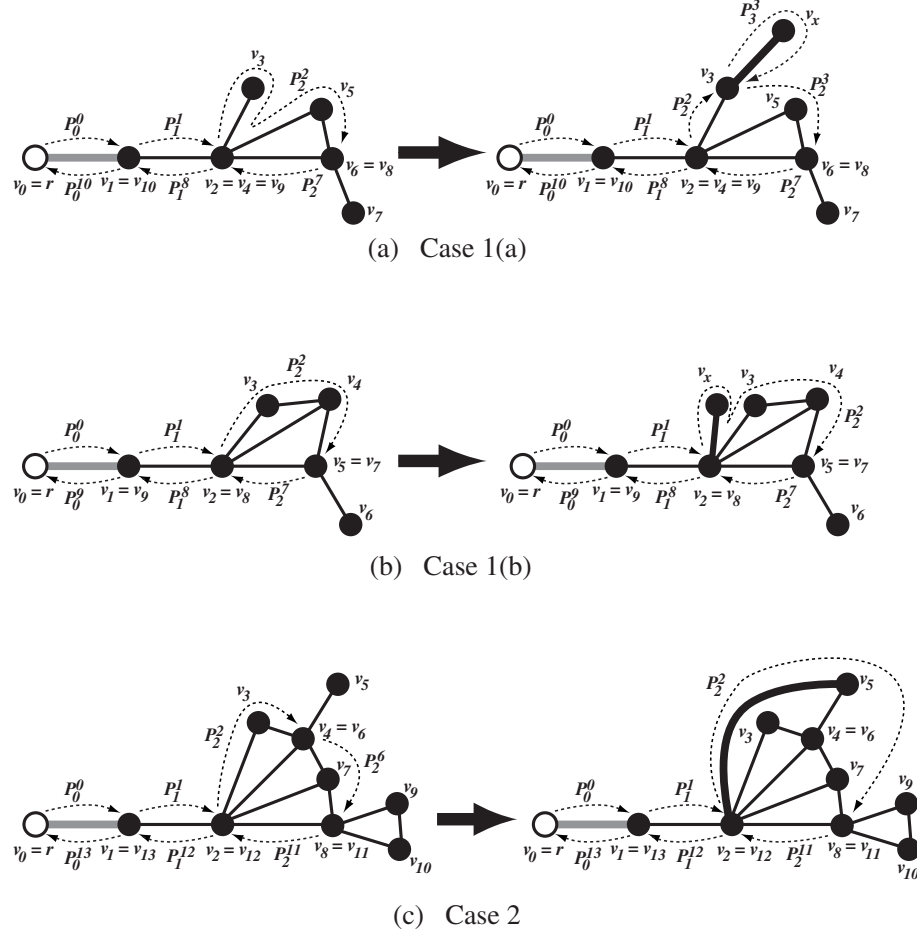


Figure 6: An update of a zombie list for (a), (b)  $G(i)$  and (c)  $G(i, j)$ .

By the above case analysis, we can prepare the zombie list of each child graph of  $G$  in  $O(1)$  time.

Next we estimate the space for zombie lists.

Since the number of dead paths of vertex  $v$  is bounded by the degree of  $v$ , the space to store the zombie lists for  $G$  is bounded by  $O(m) = O(n)$ .

By maintaining the zombie lists, if  $G$  has  $l'$  child graphs of type  $G(i, j)$ , we can find all such child graphs in  $O(l')$  time. Thus, the algorithm runs in  $O(g(m))$  time, where  $g(m)$  is the number of nonisomorphic connected rooted plane graphs with at most  $m$  edges.

## 5 Conclusion

In this paper we have given a simple algorithm to generate all connected plane graphs with at most  $m$  edges. Our algorithm first defines a family tree whose vertices correspond to graphs, then outputs each graph without duplications by traversing the tree.

By slightly modifying our algorithm with the technique in [11] one may generate all connected rooted plane graphs having exactly  $m$  edges in  $O(1)$  time for each. We need to design a family tree in which each vertex corresponds to a connected rooted plane graph with  $m$  edges.

Can we efficiently generate all “non-rooted” plane graphs? By modifying our algorithm with the technique in [8, 10] one can design an algorithm to generate all “non-rooted” plane graphs having exactly  $m$  edges in  $O(m^3)$  time for each.

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