

Announcements

- Assignment 1 (**due next Wednesday**)
- Keep checking Discussion Board on Blackboard for Assignment **Q and A**
 - **Q1, Part 3**: No formal proof required but a description of the algorithm is needed.
- Tutorial this week
 - Hierarchical Models and continuation of OpenGL for Assignment 1
- **Office Hours today 1-2 pm**

Last week's review

■ 2D Geometric Curves

- **Forms:** Explicit, Implicit, Parametric
- **Primitives:** Lines, Circles, Ellipse, Super-ellipse
- Normals and Tangents
- Polygons

■ 2D Transforms

- **Types:** Rigid, Conformal, Affine
- **Examples:** Translation, Rotation, Scaling, Shearing
- **Properties:** preserves parallelism, preserves linearity (for affine)
- **Interpretations and Uses:**
 - changing of coordinate frames
 - hierarchical models

■ Homogeneous Coordinates

Homogeneous Coordinates Review

Points

$$\bar{\mathbf{p}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \longrightarrow \hat{\mathbf{p}} = \alpha \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \mathbf{x} \\ \alpha \mathbf{y} \\ \alpha \end{bmatrix} \quad \alpha \neq 0$$

Cartesian point

Homogeneous point

Vectors

$$\hat{\mathbf{v}} = \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ 0 \end{bmatrix}$$

Homogeneous vector
(third component 0!)

$$\hat{\mathbf{p}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{x}/\alpha \\ \mathbf{y}/\alpha \\ 1 \end{bmatrix} \xrightarrow{\alpha \neq 0} \bar{\mathbf{p}} = \begin{bmatrix} \mathbf{x}/\alpha \\ \mathbf{y}/\alpha \end{bmatrix}$$

Homogeneous point

Cartesian point

$$\vec{\mathbf{v}} = \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{bmatrix}$$

Cartesian vector

Coordinate Free Geometry: Introduction to Basic Ideas

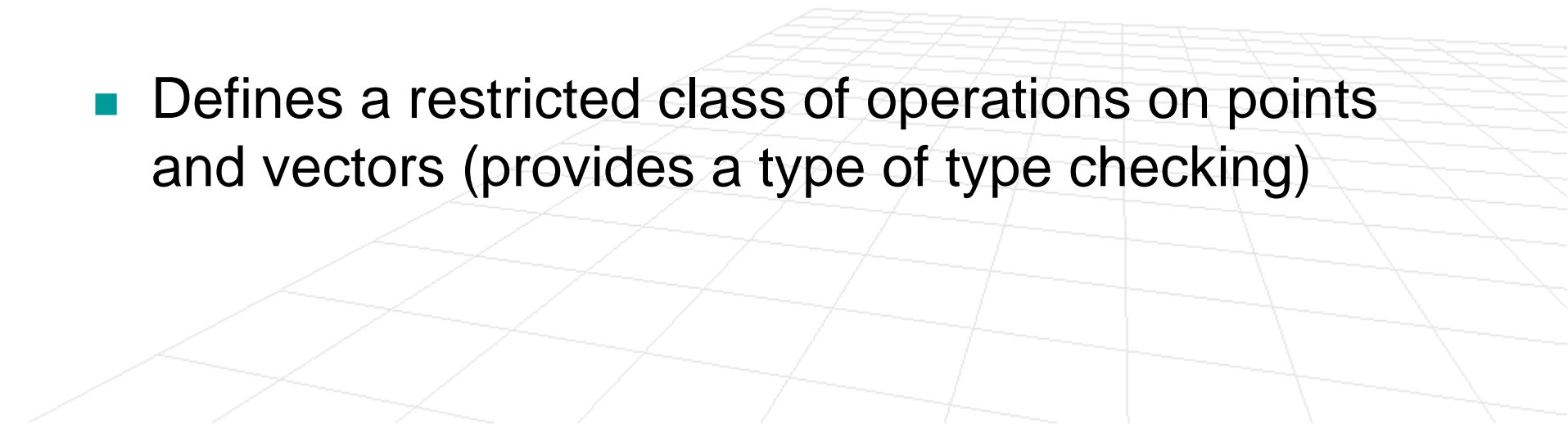
Computer Graphics, CSCD18

Fall 2008

Instructor: Leonid Sigal

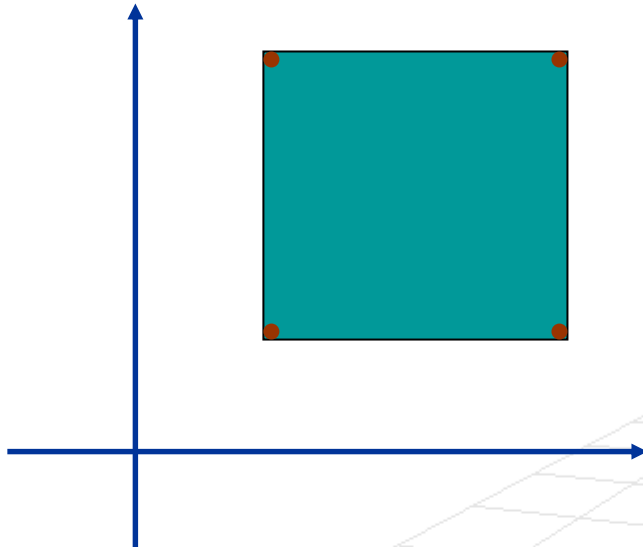


Coordinate Free Geometry

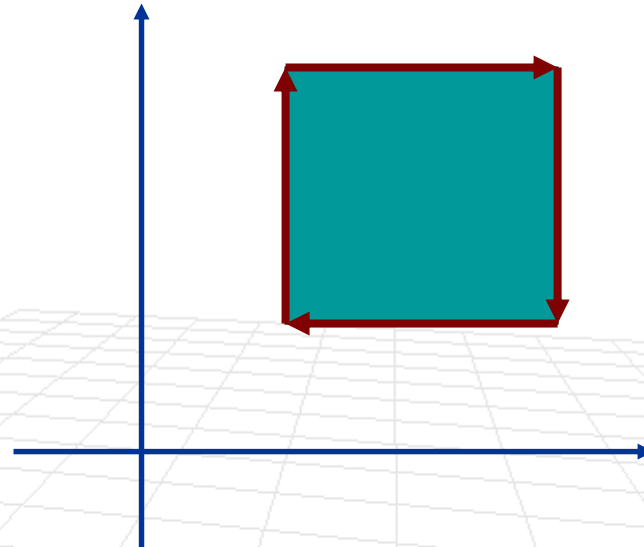
- **Coordinate Free Geometry** - style of expressing geometric objects and relations that avoids reliance on coordinate systems
 - Useful in CG where many coordinate systems are in play
 - Defines a restricted class of operations on points and vectors (provides a type of type checking)
- 

Basic Idea

**Coordinate Geometric
Representation**



**Coordinate Free Geometric
Representation**



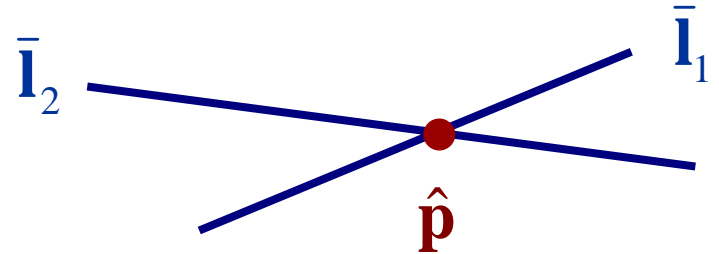
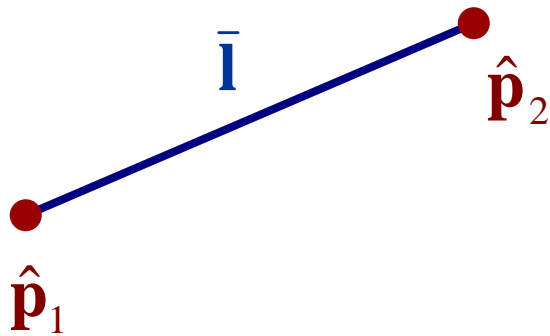
CFG Defines a set of Valid Operates on Basic Quantities

- **Scalar** – real number
- **Point** – location in space
- **Vector** – a direction and magnitude

- **Points and vectors may be represented the same but are not**
 - vector has no location in space, but point does
 - point has no magnitude, but vector does
 - we cannot add two points; we can add two vectors

CFG Style Operations with Lines

- Think back to last week



$$\bar{\mathbf{l}} = \hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_2$$

$$\hat{\mathbf{p}} = \bar{\mathbf{l}}_1 \times \bar{\mathbf{l}}_2$$

Valid CFG Operations

- point-vector addition

$$\bar{\mathbf{p}}_1 + \vec{\mathbf{v}}_1 = \bar{\mathbf{p}}_2$$

$$\vec{\mathbf{v}}_1 = \bar{\mathbf{p}}_2 - \bar{\mathbf{p}}_1$$

- vector-vector addition

$$\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_3$$

- vector scaling

$$\lambda \vec{\mathbf{v}}_1 = \vec{\mathbf{v}}_2$$

Valid CFG Operations

- magnitude of the vector

$$\mathbf{m} = \|\vec{\mathbf{v}}\|$$

- dot product

$$\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 = \|\vec{\mathbf{v}}_1\| \|\vec{\mathbf{v}}_2\| \cos(\theta)$$

angle between vectors

- cross product

$$\vec{\mathbf{v}}_1 \times \vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_3$$

where

$$\vec{\mathbf{v}}_3 \perp \vec{\mathbf{v}}_1$$

$$\vec{\mathbf{v}}_3 \perp \vec{\mathbf{v}}_2$$

$$\|\vec{\mathbf{v}}_3\| = \|\vec{\mathbf{v}}_1\| \|\vec{\mathbf{v}}_2\| \sin(\theta)$$

Valid CFG Operations

- linear combination of vectors

$$\sum_i \lambda_i \vec{v}_i = \vec{v}$$

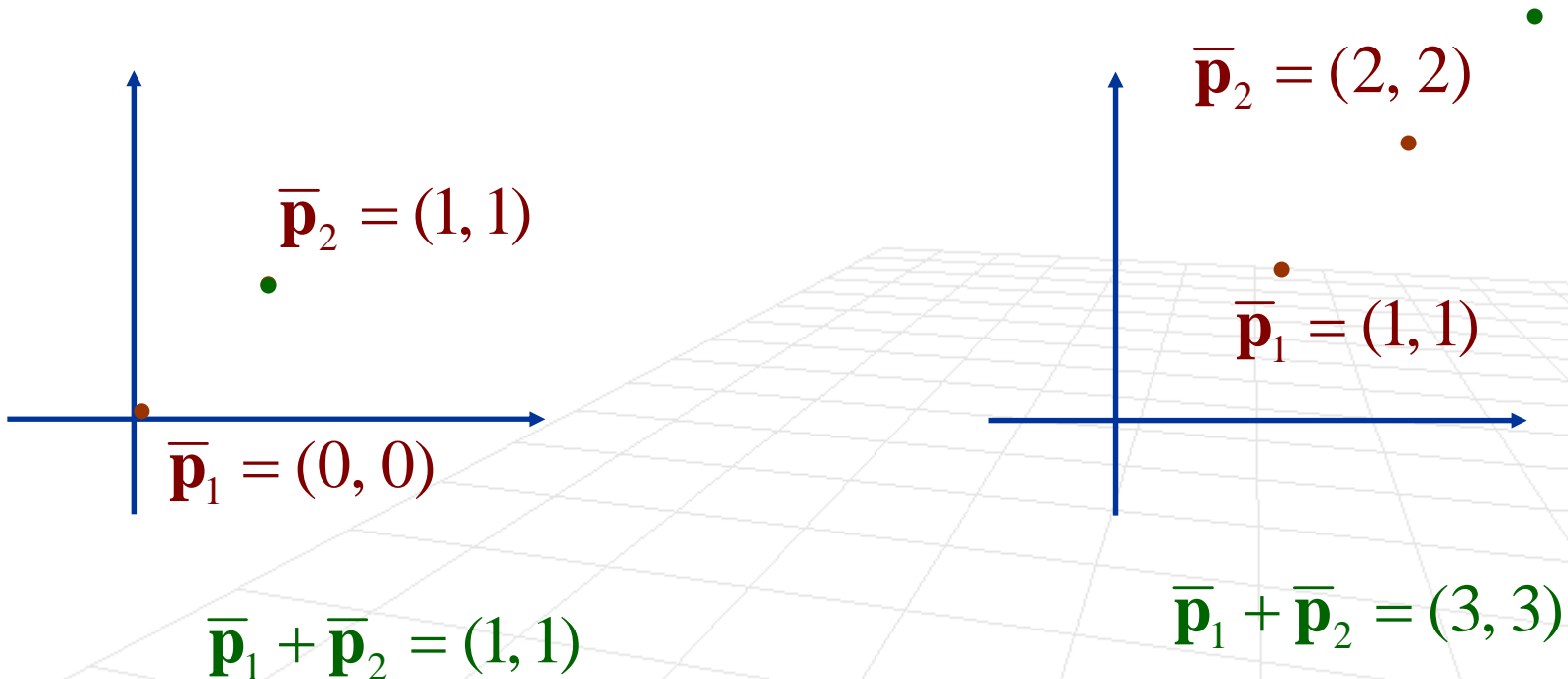
- affine combination of points

$$\sum_i \lambda_i \bar{\mathbf{p}}_i = \bar{\mathbf{p}}, \quad \sum_i \lambda_i = 1$$

$$\sum_i \lambda_i \bar{\mathbf{p}}_i = \vec{v}, \quad \sum_i \lambda_i = 0$$

Valid CFG Operations

- These are the only **valid** operations in CFG
- All other operations are **undefined**



3D Surfaces

Computer Graphics, CSCD18

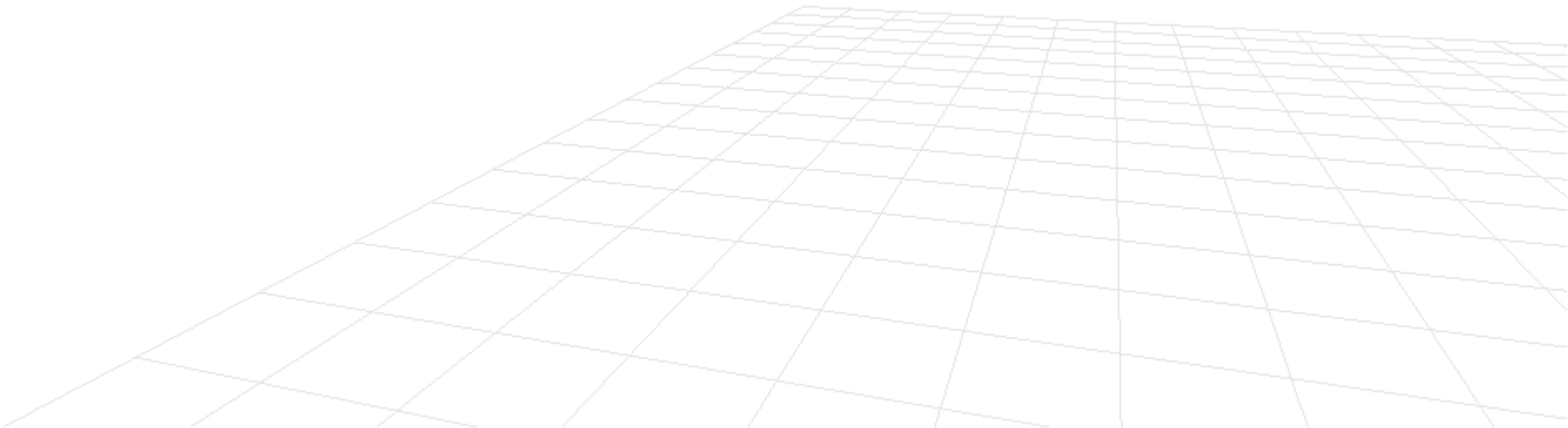
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3D surfaces

- Similar to 2D, 3D surfaces can be expressed in implicit and explicit form



Implicit Equation for the Plane

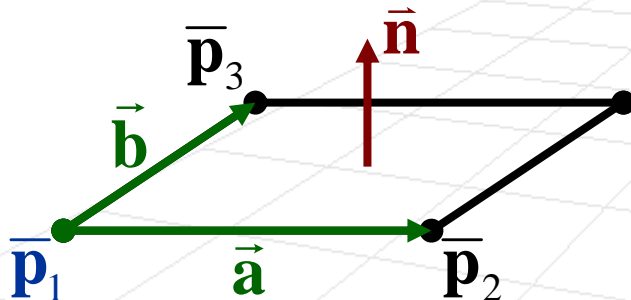
- Similar to 2D, 3D surfaces can be expressed in implicit and explicit form
- **Implicit Form:**

$$(\bar{\mathbf{p}} - \bar{\mathbf{p}}_1) \cdot \vec{\mathbf{n}} = 0$$

normal to the plane

point on a plane

Plane can be defined uniquely from 3 non-collinear points $\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_2, \bar{\mathbf{p}}_3$



$$\vec{\mathbf{a}} = \bar{\mathbf{p}}_2 - \bar{\mathbf{p}}_1$$

$$\vec{\mathbf{b}} = \bar{\mathbf{p}}_3 - \bar{\mathbf{p}}_1$$

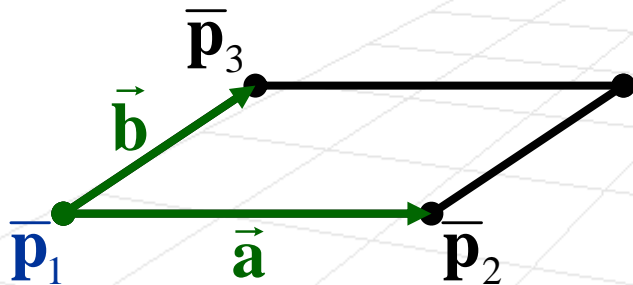
$$\vec{\mathbf{n}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$$

Parametric Equation for the Plane

- **Parametric Form:**

$$\bar{s}(\alpha, \beta) = \bar{\mathbf{p}}_1 + \alpha \vec{\mathbf{a}} + \beta \vec{\mathbf{b}} \quad \alpha, \beta \in \mathfrak{R}$$

Plane can be defined uniquely from 3 non-collinear points $\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_2, \bar{\mathbf{p}}_3$



$$\vec{\mathbf{a}} = \bar{\mathbf{p}}_2 - \bar{\mathbf{p}}_1$$

$$\vec{\mathbf{b}} = \bar{\mathbf{p}}_3 - \bar{\mathbf{p}}_1$$

Are two equations consistent?

Parametric Form:

$$\bar{\mathbf{s}}(\alpha, \beta) = \bar{\mathbf{p}}_1 + \alpha \vec{\mathbf{a}} + \beta \vec{\mathbf{b}} \quad \alpha, \beta \in \mathfrak{R}$$

Implicit Form:

$$(\bar{\mathbf{p}} - \bar{\mathbf{p}}_1) \cdot \vec{\mathbf{n}} = 0$$

Proof:

$$(\bar{\mathbf{p}}_1 + \alpha \vec{\mathbf{a}} + \beta \vec{\mathbf{b}} - \bar{\mathbf{p}}_1) \cdot \vec{\mathbf{n}} = 0$$

$$(\alpha \vec{\mathbf{a}} + \beta \vec{\mathbf{b}}) \cdot \vec{\mathbf{n}} = 0$$

$$(\alpha \vec{\mathbf{a}} + \beta \vec{\mathbf{b}}) \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) = 0$$

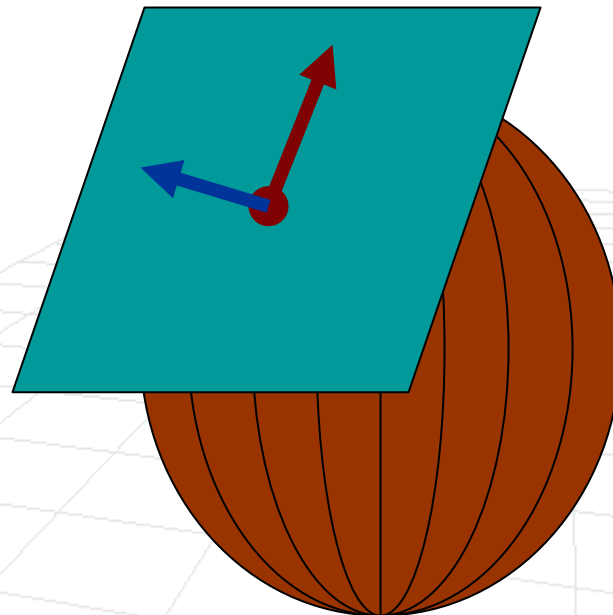
$$\alpha \vec{\mathbf{a}} \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) + \beta \vec{\mathbf{b}} \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) = 0$$

~~$\alpha \vec{\mathbf{a}} \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{b}})$~~ ~~$\beta \vec{\mathbf{b}} \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{b}})$~~ = 0

0 **0**

Surface Tangents and Normals

- **Tangent** to a surface at a point is the instantaneous direction of the surface at that point
- **Tangent plane** to surface is plane containing all tangent vectors at that point
- **Normal** to a surface is a vector perpendicular to the tangent plane



Parametric Form

- **Tangent** to a curve at a point is given by:

$$\vec{\tau}(\lambda) = \frac{d\bar{\mathbf{p}}(\lambda)}{d\lambda} = \left(\frac{d\mathbf{x}(\lambda)}{d\lambda}, \frac{d\mathbf{y}(\lambda)}{d\lambda} \right)$$

evaluated at a
given point

- The surface $\bar{\mathbf{s}}(\alpha, \beta) = (\mathbf{x}(\alpha, \beta), \mathbf{y}(\alpha, \beta), \mathbf{z}(\alpha, \beta))$ has **two tangents in a tangent plane** at a point

$$\left. \frac{\partial \bar{\mathbf{s}}(\alpha, \beta)}{\partial \alpha} \right|_{\alpha_0, \beta_0} \quad \left. \frac{\partial \bar{\mathbf{s}}(\alpha, \beta)}{\partial \beta} \right|_{\alpha_0, \beta_0}$$

- **Normal** to the surface at a point is then given by:

$$\vec{\mathbf{n}}(\alpha_0, \beta_0) = \left(\left. \frac{\partial \bar{\mathbf{s}}(\alpha, \beta)}{\partial \alpha} \right|_{\alpha_0, \beta_0} \right) \times \left(\left. \frac{\partial \bar{\mathbf{s}}(\alpha, \beta)}{\partial \beta} \right|_{\alpha_0, \beta_0} \right)$$

Parametric Form

- **Tangent plane** is then given by:

must intersect the surface at a given point

$$\vec{\mathbf{n}}(\alpha_0, \beta_0) \cdot (\bar{\mathbf{p}} - \bar{\mathbf{s}}(\alpha_0, \beta_0)) = 0$$

- The surface $\bar{\mathbf{s}}(\alpha, \beta) = (\mathbf{x}(\alpha, \beta), \mathbf{y}(\alpha, \beta), \mathbf{z}(\alpha, \beta))$ has **two tangents in a tangent plane** at a point

$$\left. \frac{\partial \bar{\mathbf{s}}(\alpha, \beta)}{\partial \alpha} \right|_{\alpha_0, \beta_0} \quad \left. \frac{\partial \bar{\mathbf{s}}(\alpha, \beta)}{\partial \beta} \right|_{\alpha_0, \beta_0}$$

- **Normal** to the surface at a point is then given by:

$$\vec{\mathbf{n}}(\alpha_0, \beta_0) = \left(\left. \frac{\partial \bar{\mathbf{s}}(\alpha, \beta)}{\partial \alpha} \right|_{\alpha_0, \beta_0} \right) \times \left(\left. \frac{\partial \bar{\mathbf{s}}(\alpha, \beta)}{\partial \beta} \right|_{\alpha_0, \beta_0} \right)$$

Implicit Form

- **Normal** to a curve at a point is given by:

$$\vec{\mathbf{n}}(\bar{\mathbf{p}}_0) = \nabla \mathbf{f}(\bar{\mathbf{p}}) \Big|_{\bar{\mathbf{p}}_0} = \left(\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{y})}{\partial \mathbf{x}} \Big|_{\bar{\mathbf{p}}_0}, \frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{y})}{\partial \mathbf{y}} \Big|_{\bar{\mathbf{p}}_0} \right)$$

- **Normal** to the surface at a point is then given by a gradient (same as in 2D)

$$\vec{\mathbf{n}}(\bar{\mathbf{p}}_0) = \nabla \mathbf{f}(\bar{\mathbf{p}}) \Big|_{\bar{\mathbf{p}}_0} = \left(\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z})}{\partial \mathbf{x}} \Big|_{\bar{\mathbf{p}}_0}, \frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z})}{\partial \mathbf{y}} \Big|_{\bar{\mathbf{p}}_0}, \frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z})}{\partial \mathbf{z}} \Big|_{\bar{\mathbf{p}}_0} \right)$$

Example: Plane

Implicit Form: $\mathbf{f}(\bar{\mathbf{p}}) = (\bar{\mathbf{p}} - \bar{\mathbf{p}}_1) \cdot \vec{\mathbf{n}} = 0$

$$\nabla \mathbf{f}(\bar{\mathbf{p}}) = \nabla (\bar{\mathbf{p}} \cdot \vec{\mathbf{n}} - \bar{\mathbf{p}}_1 \cdot \vec{\mathbf{n}})$$

not a function of p

$$\nabla \mathbf{f}(\bar{\mathbf{p}}) = \vec{\mathbf{n}}$$

Parametric Form:

$$\bar{\mathbf{s}}(\alpha, \beta) = \bar{\mathbf{p}}_1 + \alpha \vec{\mathbf{a}} + \beta \vec{\mathbf{b}} \quad \alpha, \beta \in \mathfrak{R}$$

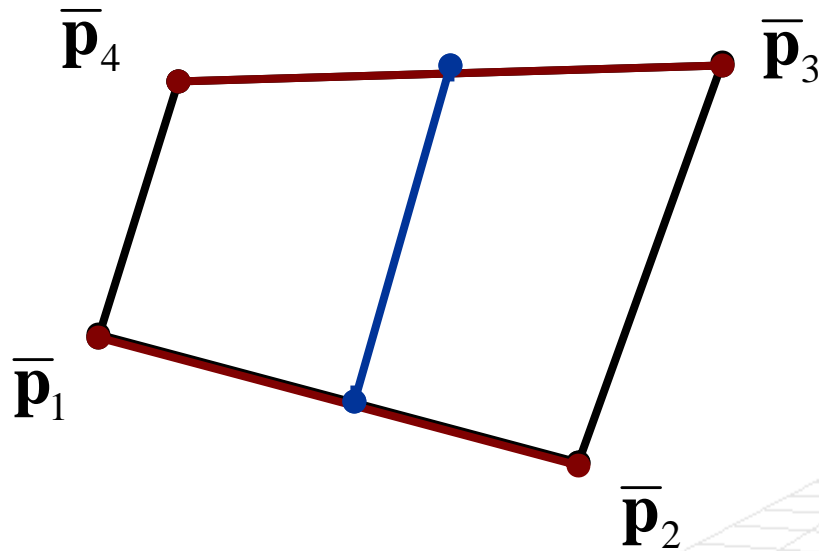
$$\frac{\partial \bar{\mathbf{s}}(\alpha, \beta)}{\partial \alpha} = \frac{\partial}{\partial \alpha} (\bar{\mathbf{p}}_1 + \alpha \vec{\mathbf{a}} + \beta \vec{\mathbf{b}}) = \vec{\mathbf{a}}$$

$$\frac{\partial \bar{\mathbf{s}}(\alpha, \beta)}{\partial \beta} = \frac{\partial}{\partial \beta} (\bar{\mathbf{p}}_1 + \alpha \vec{\mathbf{a}} + \beta \vec{\mathbf{b}}) = \vec{\mathbf{b}}$$

$$\vec{\mathbf{n}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$$

Bilinear Patch

- Defined by 4 points, no 3 of which are co-linear



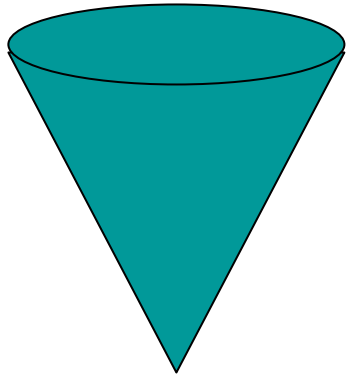
$$\bar{\mathbf{l}}_0(\alpha) = (1 - \alpha)\bar{\mathbf{p}}_1 + \alpha \bar{\mathbf{p}}_2 \quad 0 \leq \alpha \leq 1$$

$$\bar{\mathbf{l}}_1(\alpha) = (1 - \alpha)\bar{\mathbf{p}}_4 + \alpha \bar{\mathbf{p}}_3 \quad 0 \leq \alpha \leq 1$$

$$\bar{\mathbf{s}}(\alpha, \beta) = (1 - \beta)\bar{\mathbf{l}}_0(\alpha) + \beta\bar{\mathbf{l}}_1(\alpha) \quad \begin{array}{l} 0 \leq \alpha \leq 1 \\ 0 \leq \beta \leq 1 \end{array}$$

Cylinder

- Constructed by moving a point on a line along a planar curve (directrix) such that direction of line is held constant



$$\bar{\mathbf{s}}(\alpha, \beta) = \bar{\mathbf{p}}_0(\alpha) + \beta \vec{\mathbf{d}}$$

direction of the line

planar curve (directrix)

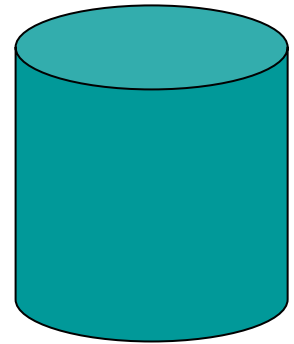
- **Right cylinder:** direction of the line perpendicular to the plane containing the curve (directrix)
- **Circular cylinder:** directrix is a circle

Example: Right Circular Cylinder

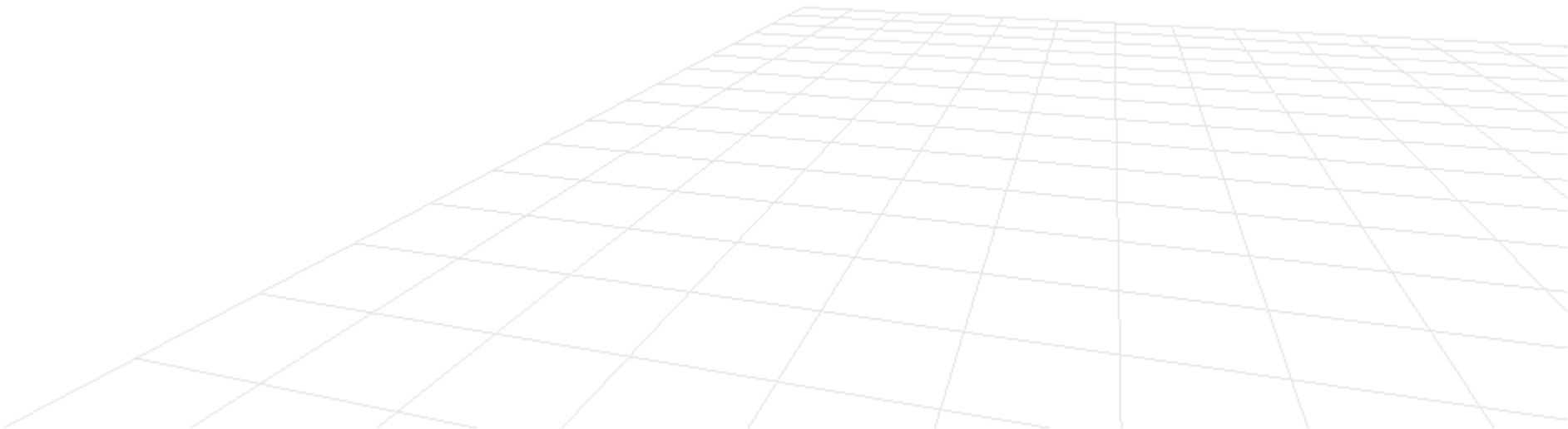
$$\bar{\mathbf{s}}(\alpha, \beta) = \bar{\mathbf{p}}_0(\alpha) + \beta \vec{\mathbf{d}}$$

$$\bar{\mathbf{p}}_0(\alpha) = (r \cos(\alpha), r \sin(\alpha), 0) \quad 0 \leq \alpha \leq 2\pi$$

$$\vec{\mathbf{d}} = (0, 0, 1)$$

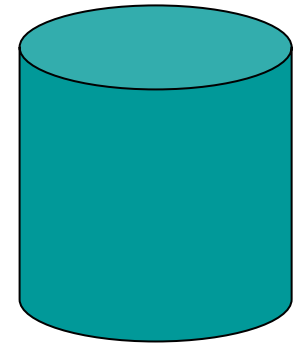


$$\bar{\mathbf{s}}(\alpha, \beta) = (r \cos(\alpha), r \sin(\alpha), \beta) \quad 0 \leq \beta \leq 1$$



Example: Right Circular Cylinder

$$\bar{\mathbf{s}}(\alpha, \beta) = (\mathbf{r} \cos(\alpha), \mathbf{r} \sin(\alpha), \beta) \quad 0 \leq \beta \leq 1$$



How do we find a normal?

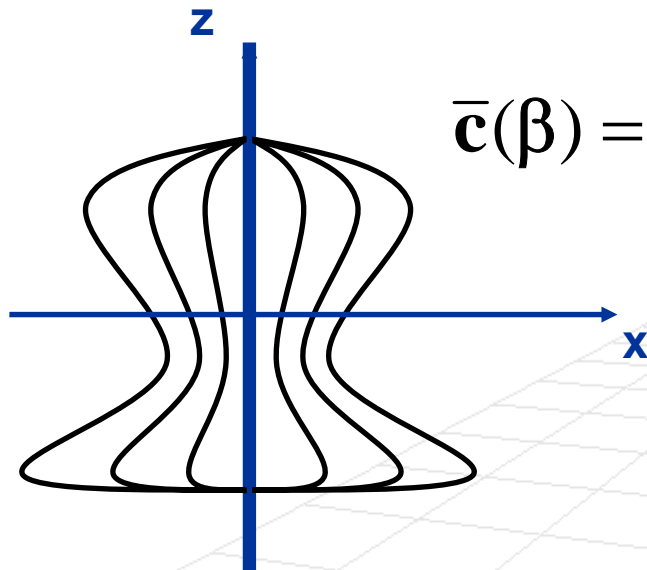
$$\frac{\partial \bar{\mathbf{s}}(\alpha, \beta)}{\partial \alpha} = (-\mathbf{r} \sin(\alpha), \mathbf{r} \cos(\alpha), 0)$$

$$\frac{\partial \bar{\mathbf{s}}(\alpha, \beta)}{\partial \beta} = (0, 0, 1)$$

$$\begin{aligned} \vec{\mathbf{n}}(\alpha, \beta) &= \frac{\partial \bar{\mathbf{s}}(\alpha, \beta)}{\partial \alpha} \times \frac{\partial \bar{\mathbf{s}}(\alpha, \beta)}{\partial \beta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\mathbf{r} \sin(\alpha) & \mathbf{r} \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= (\mathbf{r} \cos(\alpha), \mathbf{r} \sin(\alpha), 0) \end{aligned}$$

Surface of Revolution

- **Cylinder** is a special case of more general Surface of Revolution idea
- **Idea:** take any curve in x-z plane and revolve around z-axis



$$\bar{\mathbf{c}}(\beta) = (\mathbf{x}(\beta), 0, \mathbf{z}(\beta))$$

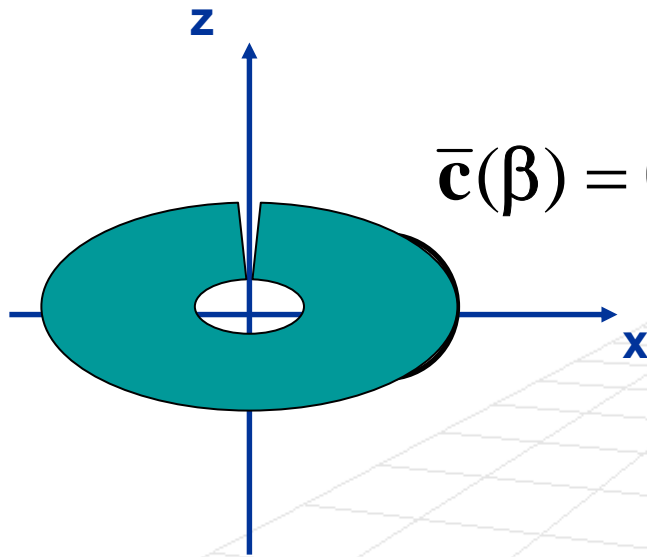
Parameter's of revolution

$$(\mathbf{r} \cos(\alpha), \mathbf{r} \sin(\alpha), 0)$$

$$\bar{\mathbf{s}}(\alpha, \beta) = (\mathbf{x}(\beta) \cos(\alpha), \mathbf{x}(\beta) \sin(\alpha), \mathbf{z}(\beta))$$

Example: Torus

- **Cylinder** is a special case of more general Surface of Revolution idea
- **Idea:** take any curve in x-z plane and revolve around z-axis



$$\bar{\mathbf{c}}(\beta) = (\mathbf{d} + \mathbf{r} \cos(\beta), 0, \mathbf{r} \sin(\beta))$$

Parameter's of revolution

$$(\mathbf{r} \cos(\alpha), \mathbf{r} \sin(\alpha), 0)$$

$$\bar{\mathbf{s}}(\alpha, \beta) = ((\mathbf{d} + \mathbf{r} \cos(\beta)) \cos(\alpha), (\mathbf{d} + \mathbf{r} \cos(\beta)) \sin(\alpha), \mathbf{r} \sin(\beta))$$