When Voters Strategize, Approval Voting Elects Condorcet Winners but Condorcet Methods can Elect Condorcet Losers

Warren Schudy

October 28, 2007

Abstract

We show that approval voting strategic equilibria are closely related to honest Condorcet Winners. There exists an approval equilibrium with a clear font-runner F and runner-up R if and only if the F is the clear Condorcet Winner and R the Condorcet runner-up. In contrast, we show that margins-based Condorcet methods can elect a Condorcet Loser with non-zero probibility when voters vote tactically. With strategic agents, approval voting is better at electing Condorcet Winners than some Condorcet methods!

It turns out these ideas were previously discovered by J.F. Laslier [6] and Burt Monroe [7].

1 Introduction

I have recently discovered a paper developed independantly from mine by J.F. Laslier that proves very similar results [6] (thanks to Steve Brams for pointing that out). If you want a more academically polished, rigorous but longer discussion of this result, see [6]. The Condorcet electing Condorcet Losers result was previously discovered by Burt Monroe [7] but oddly does not appear to have been officially published.

Voting systems were first studied in the late 1700s when Borda [1] and Condorcet [5] introduced their famous voting methods. In this paper we discuss a simple method, approval voting, that was used sporadically in the middle ages but was not studied scientifically until the 1970s. In this system, voters can vote for an unlimited number of candidates instead of a limit of 1 that plurality has.

Brams and Fishburn [2] prove that if voters have dichotomous preferences about the candidates (each voter cares about one binary issue only), rational voters will always vote sincerely and without strategy, and will always elect exactly the Condorcet winners. When voters are dichotomous, Arrow's paradox can be avoided, and strategy-proofness, independance of irrelevant alternatives, and monotonicty are all achieved with approval voting. Approval voting is arguably a perfect voting system for dichotomous voters. Of course, real populations do not have dichotomous preferences.

Brams and Fishburn [2] also discusses equilibria. Their model treats the front-runner and runner-up identically, leaving the voting of the candidates between the two unspecified. This results in examples that demonstrate unfavorable results for a Condorcet winner, or cycling, that involve only three candidates. My model assumes there are a distunguished favorite and runner-up, so rational voters who prefer the runner-up to the favorite will vote for all candidates better than the favorite, which differs from their assumption of all candidates better than the runner-up.

Brams and Sanver [4] consider stable approval outcomes. We make much stronger assumptions than they do and get stronger results.

Myerson and Weber 1993 [8] discusses equilibria for approval, plurality and Borda. Their model is more complicated than mine, allowing any number of front-runners with associated pairwise tie probabilities, and voters voting probabilistically. These differences allow their model to admit an equilibrium where a non-CW candidate with 40 percent support has a chance of winning, since the remaining 60 percent of the populance is divided about which candidate should defeat the 40 percent candidate. In my analysis, we concentrate on the special case of equilbria where there is a clear front-runner and runner-up. This special-case is commonly promoted, for example in Wikipedia's approval voting article. Contrasting the present work with [8] indicates that going beyond the front-runner and runner-up model is especially important when there is no Condorcet winner.

Brams and Fishburn [3] describe practical experience of using approval voting in professional society elections.

Range voting is a generalization of approval voting where you can give each candidate any score between 0 and 1. Optimal strategies never vote anything other than 0 or 1 in the Myerson and Weber model of voting equilibrium. Smith [9] shows that there exist models where other range strategies are strictly better and argues that there are enough honest voters to make range voting worth the complexity. We are undecided whether the extra expressiveness of range voting is worth the extra complexity of range ballots.

2 Notations and Main Result

Definition 1 *Let $X \rhd Y$ be the number of voters who prefer X to Y. Let $X \ominus Y$ be the margin of victory of X over Y in a runoff, i.e. $X \rhd Y - X \lhd Y$.*

Definition 2 *A candidate F is a Condorcet Winner iff for every other candidate X, $F \ominus X \geq 0$.*

Here we formalize the notion of a Condorcet Winner being a clear winner with a clear runner-up:

Definition 3 *Define F, R to be a Clear Condorcet Winning Pair (CCWP) iff for every other candidate X, $F \ominus X > F \ominus R > 0$.*

Clear Condorcet Winning Pair differs from Condorcet Winner in that it assumes no coincidences where two candidates are tied. If the candidates were generated randomly, this assumption would be true with high probability[1]. Strategic candidate positioning tends to cause ties, so the likelyhood of our main theorem applying to a realistic populace is unclear. However, approval voting seems to elect Condorcet winners in practice [3].

Definition 4 *A populace is opinionated iff every voter gives distinct utilities to every candidate.*

Note that an opinionate populace satisfies $X \rhd Y + Y \rhd X = n$, the number of voters and X, Y are candidates.

Definition 5 *Let $\checkmark X$ be the number of voters voting for candidate X in an election (the voters may or may not be in an equilibrium depending on context). This corresponds to the expected score concept of MW [8].*

[1] If number of voters is infinite, probability is 1

Definition 6 *A pair F, R is an Clear Approval Winning Pair (CAWP) iff there exists a Meyerson and Weber equilibrium with $\checkmark F > \checkmark R > \checkmark X$ for all other candidates X. We refer to F as the front-runner and R as the runner-up.*

Theorem 1 (Main) *Assuming an opinionated populace, a pair of candidates is a Clear Approval Winning Pair iff that pair is a Clear Condorcet Winning Pair.*

Note that Theorem 1 guarentees the existance of Meyerson-Weber equilibrium, but does not guarentee convergence to that equilbirum. The appendix shows an example of a populace where it is theoretically possible for expectations to cycle indefinately. We do not expect cycles of this sort to be a problem in practice.

3 Proof of Main Theorem

Throughout this section we assume that the populace is opinionated without stating so explicility.

One can readily see from Meyerson and Weber's equilbirum framework or common sense that if F, R is a CAWP then it is rational to vote for everyone better than the current front-runner, and also vote for the front-runner if and only if he's better than the runner-up.

Therefore:

Lemma 1 *If F, R are the front-runner and runner-up (voters expect $\checkmark F > \checkmark R > \checkmark X$) then rational voting yields:*

$$\checkmark X = \begin{cases} X \rhd R & X = F \\ X \rhd F & X \neq F \end{cases}$$

Now proof of the main theorem:

Proof Assume F, R is a CAWP. Plugging Lemma 1 into $\checkmark F > \checkmark R$ yields $F \rhd R > R \rhd F$, or $F \ominus R > 0$. Similarly $R \rhd F > X \rhd F$, so $F \ominus X > F \ominus R > 0$ and therefore F, R is a CCWP.

Conversely, assume F, R is a CCWP. We need to show that if voters vote rationally assuming $\checkmark F > \checkmark R > \checkmark X$ than their expectation will be justified.

Recall that CCWP means $F \ominus X > F \ominus R > 0$. Using the fact that the populace is opninionated, this implies that $F \rhd R > R \rhd F$ and $R \rhd F > X \rhd F$ so therefore $F \rhd R > R \rhd F > X \rhd F$. Using Lemma 1, we see that $\checkmark F = F \rhd R$, $\checkmark R = R \rhd F$ and $\checkmark X = X \rhd F$, and hence F, R is a CAWP. □

4 How Condorcet Methods can elect Condorcet Losers

Consider a populace with the following utilities:

Name	Fraction	Blue	Red	Hitler	Honest Vote
Blue	49	10	9	0	$B > R > H$
Red	49	9	10	0	$R > B > H$
Nutty1	1	0	1	10	$H > R > B$
Nutty2	1	1	0	10	$H > B > R$

With honest voting either red or blue is the Condorcet winner depending on which way noise breaks the tie. The reds could ensure victory by burying Blue, yielding: $50B = 50R$, $97R > 3H$, $51H > 49B$. Either red gets better turnout and is the Condorcet winner, or blue gets better turnout, creating a cycle, which by either margins or winning votes the weakest defeat is $50B > 50R$, so red wins.

The bluists could also try burrying red. If too many of both sides bury, then Hitler can win! If everyone buries, then the margins are: $50B > 50R$, $48R > 52H$, $51H > 49B$. With these votes Hitler is the Condorcet winner.

If defeat strength is measured by the margin of defeat, I am not aware of an equilibrium where Hitler's probability of winning is zero. However, if defeat strength is measured by winning votes, either side can pre-emptively truncate and ensure the election is fair.[2] Suppose blue truncates, yielding: $50B = 50R$, $98R > 2H$, $2H < 49B$. Now if red tries to set up a cycle, the weakest defeat will be $2H < 49B$, yielding Hitler's victory instead of Red's. The redists won't accomplish anything by burying and therefore won't try.

Unfortunately, this defense requires nearly every blue voter to truncate. If nearly every red voter buries but only some blue voters are willing to truncate, red can gain from burying.

Even if a Condorcet loser isn't elected, margin-based Condorcet methods and Borda count give victory to the major party that buries more. This interferes with having clean and honest politics. See [7] for a more formal proof of the results of this section using an extension of Myerson and Weber [8].

5 Conclusions

Approval voting is our favorite voting system. Approval voting is not much more complicated than plurality.[3] In practice, approval voting almost always elects Condorcet winners when they exist [3]. This paper provides theoretical justification for that observation.

IRV and winning-votes based Condorcet methods also reduce the two-party duopoly problem and are therefore better than plurality, but due to their complexity approval is better. We believe margin-based Condorcet Methods and Borda are *worse* than plurality because of the rewarding of dishonest voters and occasional election of Condorcet losers in equilibrium.

References

[1] J. Borda. Mémoire sur les élections au scrutin. *Histoire de l'Académie Royale des Sciences*, 1781.

[2] Brams and Fishburn. *Approval Voting* (1983)

[3] Brams and Fishburn. Going from Theory to Practice: The Mixed Success of Approval Voting. Working paper. `http://www.nyu.edu/gsas/dept/politics/faculty/brams/theory_to_practice.pdf`

[4] Brams and Sanver. Critical Strategies Under Approval Voting: Who Gets Ruled In And Ruled Out. Electoral Studies 25 (2006): 287-305. `http://www.nyu.edu/gsas/dept/politics/faculty/brams/avcritical.pdf`

[5] M. Condorcet. Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix. 1785.

[6] Jean-Francois Laslier. Strategic approval voting in a large electorate. `http://halshs.archives-ouvertes.fr/docs/00/12/17/51/PDF/stratapproval4.pdf`

[2] Thanks to Mike Ossipoff for pointing this out.

[3] For voting machines, approval is even simpler than plurality. Voting machines for plurality must recognize and discard ballots that vote for too many candidates. Approval voting eliminates this requirement.

[7] Burt Monroe. Raising Turkeys: An Extension and Devastating Application of Myerson-Weber Voting Equilibrium. Orphaned manuscript (?), `http://accuratedemocracy.com/archive/condorcet/Monroe/004004MonroeBurt.pdf`

[8] Myerson and Weber. A theory of Voting Equilibria. American Political Science Review Vol 87, No. 1. March 1993.

[9] Warren D. Smith. Non-Approval range voting style can be uniquely strategically best. Retrieved 8/11/2007.
`http://rangevoting.org/RVstrat1.html`

A Dynamics

The main body of the paper discussed voting equilibria of approval voting with a clear front-runner and runner-up. This appendix suplements the equilibria discussion with a characterization of dynamics if there is a sequence of polling rounds where voters vote using the strategy of Lemma 1. In this section we assume that there is always a clear approval front-runner and runner-up: $\checkmark F > \checkmark R > \checkmark X$. These results are in the Appendix because the assumption that all voters change their expectations immediately to match the next poll is very dubious.

Theorem 2 *If the favorite is changed by polling, the new favorite is the candidate who beats the old favorite by the largest margin, and this margin is positive.*

Proof The new favorite F' outvotes every other candidate X: $\checkmark F' > \checkmark X$. This implies that $F' \ominus F > X \ominus F$ for $X \neq F$. For F, $F' \ominus F > F \ominus R$, and for R, $F' \ominus F > R \ominus F$. If something is greater than every other member of the set, it is the maximum by definition. The positive follows from the fact that $F' \ominus F$ is greater than a number and its negation ($F \ominus R$ and $R \ominus F$). □

This allows one to construct a function giving the successor to every favorite. One can also consider this to be a directed graph where vertices are candidates and arcs are favorite progression.

Lemma 2 *If favorite is the same three times in a row, then the runner-up will be the same the second and third times.*

Proof The choice of runner-up only affects voters' decision on whether or not to vote for the favorite. Therefore, the runner-up changing will not affect who the runner-up will be next time. □

Theorem 3 *The front-runner will be the front-runner two rounds later if and only if the front-runner is the Condorcet winner.*

Proof By theorem 2, if a Condorcet Winner is the favorite no one else can become the favorite.

By Lemma 2, if a non-CW were the favorite three times in a row, the winning pair the third round would be an equilibrium, contradicting theorem 1. By Lemma 2, the favorite cannot switch from one candidate to another and then immediately back to the first. □

As a consequence of these results, if the favorite progression graph is acyclic, the favorite is guarenteed to converge to the Condorcet Winner in at most $2n$ polling rounds. The next section gives an example of a populace with a Condorcet winner but a cyclic favorite progression graph.

A.1 Cycling Example

Here we provide an example of an electorate where a Condorcet winner exists but the front-runner cycles indefinately. We will explain the *s and the last two columns later.

Candidates	Voters									Vote Counts		
	a	b	c	d	e	f	g	h	i	1st	2nd	3rd
A	*2*	*3	*1*	7	6	6	4*	4	3*	3	6	4
B	*3	*2*	*2	6*	7	5*	5	3*	4	3	6	4
C	*4	*5	*3	*3*	2*	*2*	6	6	5	6	4	3
D	*5	*4	*4	*2*	*3*	1*	7	5	6	6	4	3
E	*6	7	*5	5*	*4*	4*	*2*	2*	*1*	4	3	6
F	7	*6	6	*4*	5*	*3*	3*	*1*	2*	4	3	6
G	*1*	*1*	7	*1*	1*	7	*1*	7	7	5	5	5

A few things to notice about the electorate. Candidates A and B, C and D, and E and F are each pairwise almost identical - no voter votes another candidate between each pair. These three pairs are in a Condorcet cycle. Voters can be classified into three groups $\{abc\}$, $\{456\}$ and $\{789\}$, where all voters in the group more or less agree except for the position of candidate G.

Candidate G is clearly the Condorcet winner, since he is the first choice of a majority. Now suppose that candidates E and F are considered to be the favorite and the runner-up respectively. The candidates receiving approval are indicated by stars *before* the ranking.[4] In the next round, C will be the front-runner and D the runner-up. The third round the leading pair will be A and B, with votes are shown by stars *after* rankings.

The fourth round E and F return to the lead, completing the cycle.

[4]This example strictly speaking violates our conditions since the top two are tied. This problem is readily solvable by replacing all of the voters with 10 voters, except for 11 copies of voter 1. For simplicity we leave the example as is.