Existence of a Nash equilibrium.

Consider a game with players $\{1, 2, \ldots I\}$, where each player i has a finite nonempty set S_i of possible *pure* strategies, and a utility function $u_i : S \to R$, from the set of (pure) strategy profiles $S = \prod_i S_i$ to the reals. A *mixed* strategy is a distribution over pure strategies, leading to the notion of mixed strategy profiles and to expected utility.

A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ is a Nash equilibrium if for every player i, and every mixed strategy σ'_i, the expected utility of i for (σ'_i, σ_{-i}) is no greater than the expected utility of i for σ. Here we use the notation $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_I)$ to denote, in profile σ, the strategies of all the players other than player i.

There does not always exist a pure Nash equilibrium.

Theorem 1 *(Nash, 1951) There exists a mixed Nash equilibrium.*

Here is a short self-contained proof.

We will define a function Φ over the space of mixed strategy profiles. We will argue that that space is compact and that Φ is continuous, hence the sequence define by: $\sigma^{(0)}$ arbitrary, $\sigma^{(n)} = \Phi(\sigma^{(n-1)})$, has an accumulation point. We will argue that every fixed point of Φ must be a Nash equilibrium, hence the proof.

The space of mixed strategy profiles is clearly compact, since it can be described as:

$$\{(\alpha_i^{(s_i)}) : \forall i, \sum_{s_i \in S_i} \alpha_i^{(s_i)} = 1;\ \forall i, \forall s_i \in S_i, 0 \leq \alpha_i^{(s_i)} \leq 1\}.$$

Given a mixed strategy profile $\alpha = (\alpha_i^{(s_i)})$, the expected utility of player i is (extending the function u_i to mixed strategies)

$$u_i(\alpha) = \sum_j \sum_{s_j \in S_j} \alpha_j^{(s_j)} u^i((s_j)_j).$$

The expected utility of player i if he were to play a particular pure strategy $s \in S_i$ instead of $(\alpha_i^{(s_i)})_{s_i}$ would be

$$u_i(s, \alpha_{-i}) = \sum_{j \neq i} \sum_{s_j \in S_j} \alpha_j^{(s_j)} u^i(s, (s_j)_{j \neq i}).$$

For $s \in S_i$, let $p_i(s, \alpha) = u_i(s, \alpha_{-i}) - u_i(\alpha)$. The function Φ will modify the mixed strategy of player i by shifting some of the weight of the distribution to give more weight to the set of strategies $s \in S_i$ for which $p_i(s) > 0$, as follows: $\Phi(\alpha) = \alpha'$, with

$$\alpha_i'^{(s_i)} = \frac{\alpha_i^{(s_i)} + \max(p_i(s_i, \alpha), 0)}{1 + \sum_{s \in S_i} \max(p_i(s, \alpha), 0)}.$$

Clearly, Φ is continuous. Finally, it is easy to see that

$$\sum_{s : p_i(s, \alpha) > 0} \alpha_i'^{(s)} = \sum_{s : p_i(s, \alpha) > 0} \frac{\alpha_i^{(s)} + p_i(s, \alpha)}{1 + \sum_{s' : p_i(s', \alpha) > 0} p_i(s', \alpha)} \geq \sum_{s : p_i(s, \alpha) > 0} \alpha_i^{(s)},$$

with equality achived only if $p_i(s, \alpha) \leq 0$ for every s.

Every fixed point of Φ must have $\sum_{s : p_i(s, \alpha) > 0} \alpha_i'^{(s)} = \sum_{s : p_i(s, \alpha) > 0} \alpha_i^{(s)}$ for every i, hence must have $p_i(s, \alpha) \leq 0$ for every i and every $s \in S_i$, hence must be a Nash equilibrium. This concludes the proof of the existence of a Nash equilibrium.